Geodesics in trees of hyperbolic and relatively hyperbolic spaces

François Gautero

GEODESICS IN TREES OF HYPERBOLIC AND RELATIVELY HYPERBOLIC SPACES

FRANÇOIS GAUTERO

Abstract. We present a careful approximation of the quasi geodesics in trees of hyperbolic and relatively hyperbolic spaces. As an application we prove a dynamical and geometric combination theorem for trees of relatively hyperbolic spaces, with both Farb's and Gromov's definitions.

1. Introduction

The main part of this paper is devoted to giving a precise description of (quasi) geodesics in trees of hyperbolic and relatively hyperbolic spaces. As an application of this description we prove a *combination theorem* for such spaces. That is, a theorem giving a condition for a tree of relatively hyperbolic spaces being a relatively hyperbolic space. In [2], the authors introduce the notions of (finite) graphs of qi-embedded spaces. Assuming the Gromov hyperbolicity of the vertex spaces and the quasiconvexity of the edge spaces in the vertex spaces, they describe sufficient conditions for the universal covering of the given graph of qi-embedded spaces to be hyperbolic and then describe group-theoretic consequences. For related papers in a group-theoretic setting, see [13, 17, 15, 16]. The paper [10] gives a new proof of [2] by an approach similar to the one presented here, in the case of mapping-tori of $\mathbb{R}$-trees (i.e. 0-hyperbolic spaces) whereas [12] treats the case of mapping-tori of surface homeomorphisms: this mapping-torus case is in some sense the prototype of the "non-acylindrical" case, which is actually the case where interesting phenomena appear.

Nowadays the attention has shifted from hyperbolic spaces to *relatively hyperbolic spaces*. A notion of relative hyperbolicity was already defined by Gromov in his seminal paper [14]. Since then it has been revisited and elaborated on in many papers. Two distinct definitions now coexist. In parallel to the Gromov relative hyperbolicity, sometimes called *strong relative hyperbolicity*, there is the notion of *weak relative hyperbolicity* introduced by Farb [9] (for alternative definitions in a group-theoretic setting see Bowditch [3] or Osin [20]). In fact, it has been proved [6, 20] (see also [3]) that Gromov's definition is equivalent to Farb's definition plus an additional property, due to Farb [9]. Relatively hyperbolic spaces in the strong (that is Gromov) sense form a class encompassing Gromov hyperbolic spaces, geometrically finite orbifolds with pinched negative curvature, CAT(0)-spaces with isolated flats among many others. First combination theorems, for group-theory inclined people, in some particular (essentially acylindrical) cases have been given in the setting of relative hyperbolicity: [1], [8] or [19, 21]. One more geometric result [12] treats a particular non-acylindrical case, namely the relative hyperbolicity of mapping-tori of surface homeomorphisms. In [18] the authors give a geometric combination theorem dealing with trees of relatively hyperbolic spaces. It heavily relies

Date: September 25, 2014.

2000 *Mathematics Subject Classification.* 20F65, 57M50.

Key words and phrases. Gromov-hyperbolicity, Farb and Gromov relative hyperbolicity, trees of spaces, mapping-tori, combination theorem.

upon [2], which is used as a "black-box". In the current paper, as an application of our work on geodesics in trees of spaces we offer a quite general combination theorem for relatively hyperbolic spaces. We emphasize at once that we do not appeal to [2], but instead give a new proof of it as a particular case. Where the authors of [2] use "second-order" geometric characterization of hyperbolicity via isoperimetric inequalities, we use "first-order" geometric characterization, via approximations of geodesics and the thin triangle property. At the expense of heavier and sometimes tedious computations, this naive approach allows us to simultaneously deal with both absolute and relative hyperbolicity.

The group-theoretic consequences of the geometric combination theorem we prove here have been postponed to another paper. The first versions of this work, which go back to 2005 (and were presented in 2006 for the defense of the habilitation thesis of the author [11]), included them: geometry and group-theory were intimately linked, which at some points caused some unnecessary complications and vague formulations. R. Weidmann pointed out the needed clarifications, which lead on the one hand to a clear statement of a geometric and dynamical combination theorem (the result of the current paper), and on the other hand to a much more general group-theoretic result. This is why we chose to separate the two points of view.

Acknowledgements. Warm thanks are due to I. Kapovich (Urbana-Champaign) for his numerous explanations about the combination theorem. At that time, the author was Assistant at the University of Geneva, whereas I. Kapovich visited this university thanks to a funding of the Swiss National Science Foundation. The author is glad to acknowledge support of these institutions, as well as the support of the University Blaise Pascal (Clermont-Ferrand) where this work was finalized. Professor P. de la Harpe (Geneva) also deserves a great share of these acknowledgements for his help. Last but not least R. Weidmann (Kiel) and the referee provided an invaluable help to correct some mistakes and get a better presentation.

2. Statements of results

We begin with recalling basic definitions about coarse geometry, and in particular Gromov hyperbolicity.

A (λ, μ)-*quasi isometric embedding* from a metric space (X_1, d_1) to a metric space (X_2, d_2) is a map $f\colon X_1 \to X_2$ such that, for any x, y in X_1:

$$\frac{1}{\lambda} d_1(x,y) - \mu \leq d_2(f(x), f(y)) \leq \lambda d_1(x,y) + \mu$$

A (λ, μ)-*quasi isometry* $f\colon (X_1, d_1) \to (X_2, d_2)$ is a (λ, μ)-quasi isometric embedding such that for any $y \in X_2$ there exists $x \in X_1$ with $d_2(f(x), y) \leq \mu$.

A (λ, μ)-*quasi geodesic* in a metric space (X, d) is the image of an interval of the real line under a (λ, μ)-quasi isometric embedding.

Since quasi isometric embeddings are not necessarily continuous, a quasi geodesic as defined above is not a path in the usual sense. A *geodesic* is a $(1, 0)$-quasi geodesic. We denote by $[x, y]$ any geodesic between two points x and y in a metric space (X, d). A *geodesic space* is a metric space in which there exists (at least) one geodesic between any two points. We will need the slightly more general notion of *quasi geodesic space*: a (r, s)-*quasi geodesic space* is a metric space (X, d) in which there exists (at least) one (r, s)-quasi geodesic between any points; a *quasi geodesic space* is a metric space which is a (r, s)-quasi geodesic space for some constants $r \geq 1$ and $s \geq 0$.

We work with a version of Gromov hyperbolic spaces which is slightly extended with respect to the one most commonly used by not requiring *properness*, that is closed balls are not necessarily compact. Not requiring our spaces to be proper is important in order to deal with relatively hyperbolic spaces, the definitions of which involve non-proper metric spaces. A geodesic triangle in a metric space (X, d) is δ-*thin* if and only if any side is contained in the δ-neighborhood of the union of the two other sides. (Quasi) geodesic triangles in a (quasi) geodesic metric space (X, d) are *thin* if there exists $\delta \geq 0$ such that all (quasi) geodesic triangles are δ-thin (of course in the case of a quasi geodesic space the constant δ depends on the constants of quasi geodesicity - denoted r, s above). In this case, X is a δ-*hyperbolic space*. A metric space (X, d) is a *Gromov hyperbolic space* if and only if there exists $\delta \geq 0$ such that (X, d) is a δ-hyperbolic space.

We now recall the definitions of weak and strong relative hyperbolicity. Both notions were defined in [9]. If S is a set, the *cone with base* S is the space $S \times [0, \frac{1}{2}]$ with $S \times \{0\}$ collapsed to a point, termed the *vertex of the cone* or *cone-vertex*. This cone is considered as a metric space, with distance function $d_S((x,t),(y,t')) = t + t'$ if $x \neq y$ and $d_S((x,t),(x,t')) = |t - t'|$. Let (X, d) be a geodesic space. Putting a cone over a closed subset S of X consists of pasting to X a cone with base S by identifying $S \times \{1/2\}$ with $S \subset X$. The resulting metric space (i.e. the metric is the quotient metric - see [5][§5.18]) is denoted by $\widehat{X}$ and its subspace consisting of the cone over S by $\widehat{S}$. The space $\widehat{X}$ is such that all the points in S are now at distance $\frac{1}{2}$ from the cone-vertex and so at distance at most 1 one from each other.

Definition 2.1.
A *geodesic pair* $(X, \mathcal{P})$ is a geodesic space X equipped with a family of disjoint closed subspaces $\mathcal{P} = \{P_i\}_{i \in \Lambda}$, termed *parabolic subspaces*, which are geodesic subspaces with respect to the induced path-metric.

The induced path-metric on P_i is the path-metric obtained by defining the distance between two points in P_i as the infimum of the lengths of the paths in P_i between these two points, the length being measured with respect to the metric of X. We could only require the parabolic subspaces to be quasi geodesic subspaces, the adaptations thereafter are straightforward.

Definition 2.2. [9]
Let $(X, \mathcal{P})$ be a geodesic pair.

(a) The *coned-space* $(\widehat{X}_{\mathcal{P}}, d_{\mathcal{P}})$ is the metric space obtained from $(X, \mathcal{P})$ by putting a cone over each parabolic subspace in $\mathcal{P}$ and $d_{\mathcal{P}}$ is the *coned*, or *relative distance*.

(b) The space X is weakly hyperbolic relative to $\mathcal{P}$ if and only if the coned-space $(\widehat{X}, d_{\mathcal{P}})$ is Gromov hyperbolic.

Let $(\widehat{X}_{\mathcal{P}}, d_{\mathcal{P}})$ be a coned-space. We say that a path $\widehat{g}$ in $\widehat{X}$ *backtracks* if for the arc-length parametrization of $g\colon [0, l] \to \widehat{X}$ there exists a parabolic subspace P_i and times $0 \leq t_0 < t_1 < t_2 \leq l$ such that $g(t_0) \in P_i$, $g(t_2) \in P_i$ and $g(t_1) \notin P_i$. In other words a path backtracks if and only if it reenters a parabolic subspace that it left before. Let $\widehat{g}$ be a (u, v)-quasi geodesic path in $(\widehat{X}_{\mathcal{P}}, d_{\mathcal{P}})$ which does not backtrack. A *trace* g of $\widehat{g}$ is a subpath of X obtained by substituting each subpath of $\widehat{g}$ in the complement of X by a subpath in some parabolic subspace P_i, which is a geodesic for the path-metric induced by X on P_i.

Definition 2.3. [9] Let $(X, \mathcal{P})$ be a geodesic pair.

The coned-space $(\widehat{X}_{\mathcal{P}}, d_{\mathcal{P}})$ satisfies the *Bounded-Parabolic Penetration property (BPP)* if and only if there exists $C(u,v) \geq 0$ such that, for any two (u,v)-quasi geodesics $\widehat{g}_0, \widehat{g}_1$ of $(\widehat{X}_{\mathcal{P}}, d_{\mathcal{P}})$ with traces g_0, g_1 in (X,d), which have the same initial point, which have terminal points at most one apart and which do not backtrack, the following two properties are satisfied:

(a) if both g_0 and g_1 intersects a parabolic subspace P_i then their first intersection points with P_i are $C(u,v)$-close in (X,d),

(b) if g_0 intersects a parabolic subspace P_i and g_1 does not intersect P_i, then the diameter in (X,d) of $g_0 \cap P_i$ is bounded from above by $C(u,v)$.

Definition 2.4. [9] Let $(X, \mathcal{P})$ be a geodesic pair.

The space X is *strongly hyperbolic relative to* $\mathcal{P}$ if and only if the coned-space $(\widehat{X}_{\mathcal{P}}, d_{\mathcal{P}})$ is Gromov hyperbolic and satisfies the BPP.

Since the ultimate goal is a theorem about trees of relatively hyperbolic spaces, we introduce some notations for graphs and graphs of spaces. If Γ is a graph, $V(\Gamma)$ (resp. $E(\Gamma)$) denotes its set of vertices (resp. of oriented edges). For $e \in E(\Gamma)$ we denote by e^{-1} the same edge with opposite orientation. The map $e \mapsto e^{-1}$ is a fixed-point free involution of $E(\Gamma)$. If p is an edge-path in Γ, in particular if p is an edge, $i(p)$ (resp. $t(p)$) denotes the initial (resp. terminal) vertex of p. An edge-path p is *reduced* if no edge e in p is followed by its opposite e^{-1}. In a tree, given any two vertices x, y, we denote by $[x,y]$ the unique reduced edge-path from x to y. A *metric tree* T is a tree equipped with a length one on each edge e and an isometry from e to the real interval $(0,1)$. If p is a path in a metric tree T then $|p|_T$ denotes the length of p in T, whereas $d_T(x,y) \equiv |[x,y]|_T$ denotes the geodesic distance between any two points x, y in T.

Definition 2.5. (compare [2])

(a) A *tree of geodesic spaces* $\mathfrak{T} = (\mathcal{T}, \{X_e\}, \{X_v\}, \{\jmath_e\})$ is a metric tree $\mathcal{T}$ with length 1 edges, together with two collections of geodesic spaces, the collection of *edge-spaces* $\{X_e\}_{e \in E(\Gamma)}$ indexed over the oriented edges e of $\mathcal{T}$ which satisfy $X_e = X_{e^{-1}}$ and the collection of *vertex-spaces* $\{X_v\}_{v \in V(\Gamma)}$ indexed over the vertices v of $\mathcal{T}$, and a collection of maps $\jmath_e \colon X_e \to X_{t(e)}$ from the edge-spaces to the vertex-spaces.

(b) A *tree of qi-embedded geodesic spaces* is a tree of geodesic spaces $(\mathcal{T}, \{X_e\}, \{X_v\}, \{\jmath_e\})$ such that there exist two fixed real constants $\mathfrak{a} \geq 1$ and $\mathfrak{b} \geq 0$ such that the maps $\jmath_e \colon X_e \to X_{t(e)}$ from the edge-spaces X_e to the vertex-spaces X_v are $(\mathfrak{a}, \mathfrak{b})$-quasi isometric embeddings.

(c) A *tree of hyperbolic spaces* is a tree of qi-embedded geodesic spaces such that there is $\delta \geq 0$ for which each edge- and vertex-space is a δ-hyperbolic space.

Of course, we could only require that the edge- and vertex-spaces be quasi geodesic spaces instead of geodesic ones, the adaptations are once again straightforward. Before defining trees of relatively hyperbolic spaces we need to introduce the notion of the *coned-extension* of a map between geodesic pairs.

Definition 2.6. Let $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ be two geodesic pairs.

(a) A map $f \colon X \to Y$ is a *pair-map from* $(X, \mathcal{P})$ *to* $(Y, \mathcal{Q})$ if and only if for every parabolic subspace $P \in \mathcal{P}$ there is a unique parabolic subspace $Q \in \mathcal{Q}$ such that $f(P) \subset Q$.

(b) Let $f \colon (X, \mathcal{P}) \to (Y, \mathcal{Q})$ be a pair-map and let $\widehat{X}, \widehat{Y}$ be the coned-spaces associated to $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ respectively. A map $\widehat{f} \colon \widehat{X} \to \widehat{Y}$ is a *coned-extension of* f if and only if it satisfies the following properties:

- Its restriction to X is equal to f.
- For any parabolic subspace $P \in \mathcal{P}$ with $f(P) \subset Q \subset \widehat{Q}$, $\widehat{f}$ is a pair-map from $(\widehat{X}, \widehat{P} \setminus P)$ to $(\widehat{Y}, \widehat{Q} \setminus Q)$ which sends the cone-vertex of $\widehat{P}$ to the cone-vertex of $\widehat{Q}$.

Definition 2.7.

(a) A *tree of geodesic pairs* $(\mathcal{T}, \{(X_e, \mathcal{P}_e)\}, \{(X_v, \mathcal{P}_v)\}, \{\jmath_e\})$ is a tree of geodesic spaces $(\mathcal{T}, \{X_e\}, \{X_v\}, \{\jmath_e\})$ such that for each edge e and each vertex v, $(X_e, \mathcal{P}_e)$ and $(X_v, \mathcal{P}_v)$ are geodesic pairs, for each edge e, $\mathcal{P}_e = \mathcal{P}_{e^{-1}}$ and $\jmath_e \colon (X_e, \mathcal{P}_e) \to (X_{t(e)}, \mathcal{P}_{t(e)})$ is a pair-map.

(b) A *tree of weakly* (resp. *strongly*) relatively hyperbolic spaces is a tree of geodesic pairs $\mathfrak{T} = (\mathcal{T}, \{(X_e, \mathcal{P}_e)\}, \{(X_v, \mathcal{P}_v)\}, \{\jmath_e\})$ such that:

- For each edge e, the edge-space X_e is weakly (resp. strongly) hyperbolic relatively to the family of parabolic subspaces $\mathcal{P}_e$. For each vertex v the vertex-space X_v is weakly (resp. strongly) hyperbolic relative to the family of parabolic subspaces $\mathcal{P}_v$.
- If $\widehat{X}_e$ and $\widehat{X}_v$ denote the coned-spaces equipped with the relative metrics associated to the geodesic pairs $(X_e, \mathcal{P}_e)$ and $(X_v, \mathcal{P}_v)$ and $\widehat{\jmath}_e$ is a coned-extension of $\jmath_e$ then $\widehat{\mathfrak{T}} = (\mathcal{T}, \{\widehat{X}_e\}, \{\widehat{X}_v\}, \{\widehat{\jmath}_e\})$ is a tree of qi-embedded geodesic spaces.

Remark 2.8. Our definition is more general than the corresponding definition in [18] because we do not require that the attaching-maps of the edge-spaces to the vertex-spaces be quasi isometric embeddings for the absolute metrics but only for the relative metrics.

Definition 2.9. Let $\mathfrak{T} = (\mathcal{T}, \{X_e\}, \{X_v\}, \{\jmath_e\})$ be a tree of geodesic spaces.

If $E^+(\mathcal{T})$ denotes the subset of $E(\mathcal{T})$ composed of exactly one representative in each pair (e, e^{-1}) then the space $\widetilde{X}$ obtained from

$$\bigsqcup_{e \in E^+(\mathcal{T})} (X_e \times [0,1]) \sqcup \bigsqcup_{v \in V(\mathcal{T})} X_v$$

by identifying $(x, 1) \in X_e \times [0,1]$ with $\jmath_e(x) \in X_{t(e)}$ and $(x, 0) \in X_e \times [0,1]$ with $\jmath_{e^{-1}}(x) \in X_{i(e)}$ for each $e \in E^+(\mathcal{T})$ is called the *geometric realization of* $\mathfrak{T}$.

We denote by $\pi \colon \widetilde{X} \to \mathcal{T}$ the map which identifies each subset $X_e \times \{t\} \subset \widetilde{X}$ with the point in $e \in E(\mathcal{T})$ with coordinate $t \in [0,1]$ (recall that each edge e comes with an isometry with $[0,1]$) and each subset $X_v \subset \widetilde{X}$ with the vertex v of $\mathcal{T}$. The sets $X_e \times \{t\}$ with $t \in (0,1)$ and X_v are the *strata* of $\widetilde{X}$. A path contained in a stratum is a *horizontal path*. By definition, each stratum in a tree of qi-embedded geodesic spaces comes with a metric. The associated length function defined on horizontal paths is termed *horizontal length* and the horizontal length of a horizontal path is denoted by $|p|_{hor}$. Similarly, the distance function associated to the metric of a stratum, which is defined for any two points in this stratum, is termed *horizontal distance* and the horizontal distance between any two points x, y in a same stratum is denoted by $d_{hor}(x, y)$. Each subset $\{x\} \times [0,1]$, $x \in X_e$ for some edge e, also has its natural metric, the usual metric on $[0,1]$, which gives the notion of *interval-length* for subpaths contained in such subsets.

Definition 2.10. Let $(\widetilde{X}, \pi, \mathcal{T})$ be the geometric realization of a tree of qi-embedded geodesic spaces. For any two points x, y in $\widetilde{X}$, let $\mathcal{P}(x, y)$ be the set of all the continuous paths from x to y which are the concatenation of horizontal paths and of non-trivial intervals (that is intervals not degenerate to a point).

The *tree of spaces-distance* between any two points x, y in $\widetilde{X}$, denoted by $d_{\widetilde{X}}(x,y)$, is the infimum of the lengths of the paths in $\mathcal{P}(x,y)$, measured as the sum of the horizontal and interval-lengths of their subpaths.

This tree of spaces-distance is reminiscent of the quotient-metric of [5][§5.18]. The following lemma is obvious:

Lemma 2.11. *With the notations of Definition 2.10, the space $\widetilde{X}$ equipped with the tree of spaces-distance $d_{\widetilde{X}}$ is a quasi geodesic metric space.*

Remark 2.12. The geometric realization of a tree of geodesic spaces is the space we will work with. Thus, with a slight abuse of terminology we will often denote by $(\widetilde{X},\pi,\mathcal{T})$ a tree of qi-embedded geodesic spaces and write "a tree of geodesic spaces ..." for "the geometric realization of a tree of geodesic spaces ...".

A *section of a map* $\pi\colon A\to B$ is a map $\sigma\colon B\to A$ such that $\pi\circ\sigma=\mathrm{Id}_B$ (this is only a set-theoretic notion, for instance we do not require that a section of a continuous map be continuous).

Definition 2.13. Let $(\widetilde{X},\mathcal{T},\pi)$ be the geometric realization of a tree of qi-embedded geodesic spaces.

For $v\geq 0$, a *v-vertical segment* in $\widetilde{X}$ is a section σ_ω of π over a geodesic ω of $\mathcal{T}$ which is a $(v+1,v)$-quasi isometric embedding of ω in $(\widetilde{X},d_{\widetilde{X}})$.

The *vertical length* of the v-vertical segment $\sigma_\omega\colon\omega\to\widetilde{X}$ is the length $|\omega|_{\mathcal{T}}$.

We will not distinguish a vertical segment, which by definition is a map, from its image in the tree of spaces. Since a section is not necessarily continuous, this image is of course not a segment in the usual sense. But if $\omega=e_{i_1}^{\epsilon_1}\cdots e_{i_k}^{\epsilon_k}$ is a geodesic edge-path then a v-vertical segment over ω can be approximated by a sequence of intervals $x_i\times(0,1)$ over the e_i's, the Hausdorff distance between the v-vertical segment and these intervals only depending on v.

The "hallways-flare property" was introduced in [2]: it designated the main property introduced by the authors for the hyperbolicity of a graph of quasi isometrically embedded hyperbolic spaces. Our presentation here being very different and more dynamical in nature, we use the denomination of *exponential-separation property* for our central property given in Definition 2.14 below and invite the reader to compare with the "hallways-flare property" of [2].

Definition 2.14. (compare [2])

A tree of qi-embedded spaces satisfies the *exponential-separation property* if and only if for any $v\geq 0$ there exist $\lambda>1$ and positive integers t_0, M such that, for any geodesic segment $[\beta,\gamma]\subset\mathcal{T}$ of length $2t_0$ and midpoint α, any two v-vertical segments s_0,s_1 over $[\beta,\gamma]$ with $d_{hor}(s_0\cap X_\alpha, s_1\cap X_\alpha)\geq M$ satisfy:

$$\max(d_{hor}(s_0\cap X_\beta, s_1\cap X_\beta), d_{hor}(s_0\cap X_\gamma, s_1\cap X_\gamma))\geq\lambda d_{hor}(s_0\cap X_\alpha, s_1\cap X_\alpha).$$

The constants λ, M, t_0 will be referred to as the *constants of hyperbolicity.*

We will sometimes say that the v-vertical segments are *exponentially separated.*

Theorem 2.15. *Let $\mathfrak{T}=(\mathcal{T},\{(X_e,\mathcal{P}_e)\},\{(X_v,\mathcal{P}_v)\},\{\jmath_e\})$ be a tree of weakly relatively hyperbolic spaces. If $\widehat{\mathfrak{T}}$ satisfies the exponential-separation property then $\mathfrak{T}$ is weakly hyperbolic relative to the family composed of all the parabolic subspaces of the vertex-spaces.*

Remark 2.16. In the setting of weak relative hyperbolicity we could drop the assumption that the attaching-maps $\jmath_e$ be pair-maps from $(X_e, \mathcal{P}_e)$ to $(X_{t(e)}, \mathcal{P}_{t(e)})$. In this case the statement of theorem 2.15 has to be modified by adding the collection of all the parabolic subspaces of the edge-spaces in the given family of parabolic subspaces for the tree of weakly relatively hyperbolic spaces. This follows from the proof of Theorem 2.15.

Definition 2.17. Let $(\mathcal{T}, \{(X_e, \mathcal{P}_e)\}, \{(X_v, \mathcal{P}_v)\}, \{\jmath_e\})$ be a tree of geodesic pairs.

The *induced forest of parabolic spaces* is the forest of spaces $(\mathcal{F}_{\mathcal{P}}, \{\mathfrak{P}_e\}, \{\mathfrak{P}_v\}, \{\imath_e\})$ defined as follows:

(a) There is a bijection σ_E (resp. σ_V) from the set of edges (resp. vertices) of $\mathcal{F}_{\mathcal{P}}$ to the set of all the parabolic subspaces of the edge-spaces (resp. vertex-spaces) of $\mathcal{T}$.

(b) The edge-space $\mathfrak{P}_e$ (resp. vertex-space $\mathfrak{P}_v$) of $\mathcal{F}_{\mathcal{P}}$ is the parabolic subspace $\sigma_E(e)$ (resp. parabolic subspace $\sigma_V(v)$) of $\mathcal{T}$.

(c) There is an oriented edge e with terminal vertex v in $\mathcal{F}_{\mathcal{P}}$ if and only if, letting e' be the oriented edge of $\mathcal{T}$ such that $\sigma_E(e) \subset X_{e'}$ and v' the vertex of $\mathcal{T}$ such that $\sigma_V(v) \subset X_{v'}$, one has $v' = t(e')$ and $\jmath_{e'}(\sigma_E(e)) \subset \sigma_V(v)$. In this case $\imath_e$ is the restriction of $\jmath_{e'}$ to $\sigma_E(e)$.

An *induced tree of parabolic spaces* is any connected component of the induced forest of parabolic spaces.

Remark 2.18. The geometric realization of the induced forest of parabolic spaces of a tree of geodesic pairs is naturally embedded in the geometric realization of the latter. So, assimilating this forest and the tree to their geometric realizations, it makes sense to speak about the "horizontal distance between two induced trees of parabolic spaces" or about the vertical diameter of some of their subsets.

Definition 2.19. A tree of strongly relatively hyperbolic spaces satisfies the *strong exponential-separation property* if and only if it satisfies the exponential-separation property and for any $l \geq 0$ there is $t \geq 0$ such that for any two distinct induced trees of parabolic spaces, the union of all the strata where they are at horizontal distance smaller than l has vertical diameter smaller than t.

Theorem 2.20. *Let $\mathfrak{T} = (\mathcal{T}, \{(X_e, \mathcal{P}_e)\}, \{(X_v, \mathcal{P}_v)\}, \{\jmath_e\})$ be a tree of strongly relatively hyperbolic spaces. If $\widehat{\mathfrak{T}}$ satisfies the strong exponential-separation property then $\mathfrak{T}$ is strongly hyperbolic relatively to the family composed of all the induced trees of parabolic spaces.*

Remark 2.21. In Theorems 2.15 and 2.20, the assumption that the v-vertical segments are exponentially separated for any $v \geq 0$ could be substituted by the weaker assumption that the v-vertical segments be exponentially separated for some constant v sufficiently large.

2.1. **Plan of the paper:** The results above are consequences of Theorems 4.6 and 5.2 about the behavior of quasi geodesics in trees of hyperbolic spaces. Section 3 contains some technical consequences of the basic notions exposed above. Section 4 deals with the approximation of quasi geodesics in the particular case where all the attaching-maps of the considered tree of hyperbolic spaces are quasi isometries. Section 5 contains the adaptations to the general case. The important notions appearing in these two sections are the corridors in Section 4, and the generalized corridors in Section 5. These two sections appeal to two important Propositions whose proofs are delayed: Proposition 4.7 is proved in Section 8; Proposition 4.8 is proved in Section 9 whereas its adaptation to generalized corridors (Proposition 5.4) is dealt with in subsection 9.6. In Section 6 the reader will find the proof of Theorem 2.15 (weak relative hyperbolicity case) whereas Section 7 deals with the proof of Theorem 2.20 (strong relative hyperbolicity case). This last section also contains Proposition 7.4 whose proof is postponed to subsection 9.7.

3. Preliminaries

If (X,d) is a metric space with distance function d, and x a point in X, we set $B_x(r) = \{y \in X \; ; \; d(x,y) \leq r\}$. If A and B are any two subsets of (X,d), $d(A,B) = \inf_{x\in A, y\in B} d(x,y)$. We set also $\mathcal{N}_d^r(A) = \{x \in X \; ; \; d(x,A) \leq r\}$ and $d^H(A,B) = \inf\{r \geq 0 \; ; \; A \subset \mathcal{N}_d^r(B)$ and $B \subset \mathcal{N}_d^r(A)\}$. The latter is the usual Hausdorff distance between A and B. Finally, $\mathrm{diam}_X(A)$ stands for the *diameter of* A: $\mathrm{diam}_X(A) = \sup\{d(x,y) \; ; \; x,y \in A\}$.

From now on, unless otherwise specified, the trees of qi-embedded spaces are equipped with the tree of spaces-distance $d_{\widetilde{X}}(.,.)$ introduced in Definition 2.10. This metric is a particular case of the telescopic metrics we define below. We recall that we also defined the horizontal distance, denoted by $d_{hor}(.,.)$, for pair of points belonging to a same stratum, and the horizontal length, denoted by $|.|_{hor}$ for horizontal paths, that is paths contained in a stratum. We adopt the convention that the horizontal distance is infinite for two points not belonging to a same stratum.

3.1. The telescopic metric.

Definition 3.1. Let $(\widetilde{X}, \mathcal{T}, \pi)$ be a tree of qi-embedded geodesic spaces, and let $v \geq 0$.

(a) A *v-telescopic chain* is an ordered sequence $(h_0, s_0, h_1, \cdots, h_{k-1}, s_{k-1}, h_k)$ of horizontal paths h_j and of v-vertical segments s_j in $\widetilde{X}$ such that:
- for any $k \geq j \geq 0$, h_j belongs to a vertex-space,
- for any $k-1 \geq j \geq 0$, $t(h_j) = i(s_j)$ and $t(s_j) = i(h_{j+1})$.

(b) The *vertical length* $|p|_{vert}^v$ of a v-telescopic chain $p = (h_0, s_0, h_1, \cdots, h_{k-1}, s_{k-1}, h_k)$ is equal to $\displaystyle\sum_{j=0}^{k-1} |s_j|_{vert}$.

(c) The *horizontal length* $|p|_{hor}^v$ of a v-telescopic chain $p = (h_0, s_0, h_1, \cdots, h_{k-1}, s_{k-1}, h_k)$ is equal to $\displaystyle\sum_{j=0}^{k} |h_j|_{hor}$.

(d) The *telescopic length* $|p|_{tel}^v$ of a v-telescopic chain p is equal to $|p|_{hor}^v + |p|_{vert}^v$.

(e) The *v-telescopic distance* $d_{tel}^v(x,y)$ between any two points x and y in $\widetilde{X}$ is the infimum of the telescopic lengths of the v-telescopic chains between x and y.

Remark 3.2. Setting $v = 0$ in Definition 3.1 above we get the tree of spaces-distance (compare with Definition 2.10).

Let $v \geq 0$. The definition of a v-telescopic chain (item (a) of Definition 3.1) implies in particular that for any $k-1 \geq j \geq 0$, $\pi(s_0)\pi(s_1)\cdots\pi(s_j)$ is an edge-path between two vertices of $\mathcal{T}$. Any non-trivial (i.e. not degenerate to a point) v-vertical segment in a v-telescopic chain has v-vertical length greater or equal to one. For any $x \in \widetilde{X}$ there is $w \in V(\mathcal{T})$ such that $d_{vert}^v(x, \pi^{-1}(w)) \leq \frac{1}{2}$. It follows from the latter observation that, when dealing with the behavior of quasi geodesics or with the hyperbolicity of $\widetilde{X}$, there is no harm in requiring that telescopic chains begin and end at strata over vertices of $\mathcal{T}$, as was done in Definition 3.1.

For the sake of simplification, we will often forget the superscripts in the vertical, horizontal and telescopic lengths, unless some ambiguity might exist.

Lemma 3.3. *Let $(\widetilde{X}, \mathcal{T}, \pi)$ be a tree of hyperbolic spaces.*

(a) For any $v \geq 0$ there exist $\lambda_+ \equiv \lambda_+(v) \geq 1$, $\mu \equiv \mu(v) \geq 0$ such that, if ω_0 and ω_1 are any two v-vertical segments, with initial (resp. terminal) points x_0, x_1 (resp. y_0, y_1) and such that $\pi(\omega_0) = \pi(\omega_1) = [a, b]$ then:

$$\frac{1}{\lambda_+^{d_{\mathcal{T}}(a,b)}} d_{hor}(x_0, x_1) - \mu \leq d_{hor}(y_0, y_1) \leq \lambda_+^{d_{\mathcal{T}}(a,b)} d_{hor}(x_0, x_1) + \mu.$$

The constants λ_+, μ will be referred to as the constants of quasi isometry*.*

(b) For any sequence of points $(x_n)_{n \in \mathbb{Z}^+}$ in some stratum, $\lim_{n \to +\infty} d_{hor}(x_0, x_n) = +\infty \Leftrightarrow \lim_{n \to +\infty} d^v_{tel}(x_0, x_n) = +\infty$.

(c) For any $v, v' \geq 0$, there exist $A \geq 1$, $B \geq 0$ such that the identity-map from $(\widetilde{X}, d^v_{tel})$ to $(\widetilde{X}, d^{v'}_{tel})$ is a (A, B)-quasi isometry.

(d) For any $d, v \geq 0$ there exists $C \equiv C(d, v) \geq 0$ such that for any $\alpha, \beta \in \mathcal{T}$ with $d_{\mathcal{T}}(\alpha, \beta) = d$, for any $x, y, z \in X_\alpha$ with $z \in [x, y]$, for any $x', y', z' \in X_\beta$ which are the endpoints of v-vertical segments starting respectively at x, y and z, we have $z' \in \mathcal{N}^C_{hor}([x', y'])$. Moreover, for any $d' \geq d$ and $v' \geq v$, $C(d', v') \geq C(d, v)$.

(e) For any $v, w \geq 0$, there is $D \geq 0$ such that if s is a v-vertical segment, then s is a (D, D)-quasi geodesic for the w-telescopic distance.

(f) Any concatenation of v-vertical segments $s_1, \cdots, s_n$ over reduced edge-paths p_j in $\mathcal{T}$ with $t(s_j) = i(s_{j+1})$ for $j = 1, \cdots, n-1$ and with $p_1 p_2 \cdots p_n$ being a reduced edge-path, is a $2v$-vertical segment.

Proof. There is $\rho(v)$ such that each ω_i is at Hausdorff horizontal distance at most $\rho(v)$ of a vertical segment ω'_i which is a sequence of intervals $x^j_i \times I$ over the edges e_j in $[a, b]$, where $I = (0, 1)$ for any $e_j \subset [a, b]$, $I = (\epsilon, 1]$ if $e_j \cap [a, b] = [a, t(e_j)]$ and $I = (0, \epsilon]$ if $e_j \cap [a, b] = (i(e_j), b]$, with a jump of at most $\theta(v)$ with respect to the horizontal distance in each vertex-space intersected. Each edge-space is $(\mathfrak{a}, \mathfrak{b})$-quasi isometrically embedded into the vertex-spaces (see Definition 2.5). This implies for each edge e_j

$$d_{hor}(\omega'_0 \cap X_{t(e_j)}, \omega'_1 \cap X_{t(e_j)}) \leq \mathfrak{a}(\mathfrak{a} d_{hor}(\omega'_0 \cap X_{i(e_j)}, \omega'_1 \cap X_{i(e_j)}) + \mathfrak{b}) + \mathfrak{b} + 2\theta(v).$$

Since the v-vertical segments ω'_i are at horizontal distance smaller than $\rho(v)$ from the ω_i we get

$$d_{hor}(\omega_0 \cap X_{t(e_j)}, \omega_1 \cap X_{t(e_j)}) \leq \mathfrak{a}(\mathfrak{a} d_{hor}(\omega_0 \cap X_{i(e_j)}, \omega_1 \cap X_{i(e_j)}) + \mathfrak{b}) + \mathfrak{b} + 2\theta(v) + 4\rho(v).$$

Item (a) follows by composing the inequalities given by all the edges: it suffices to set $\lambda_+ = \mathfrak{a}^2 + 1$ and $\mu = (\mathfrak{a}^2 + 1)(\mathfrak{a}\mathfrak{b} + \mathfrak{b} + 2\theta(v) + 4\rho(v))$.

If $d^v_{tel}(x_0, x_n)$ does not tend toward infinity with n, then both the horizontal lengths and the vertical length of the quasi geodesic telescopic chains between x_0 and x_n are bounded above by some constant M. Item (a) then gives $\lambda_+(v)$ and $\mu(v)$ such that $d_{hor}(x_0, x_n)$ is bounded above by $\lambda_+^M(v) M + \mu(v)$. Conversely, if $d_{hor}(x_0, x_n)$ does not tend toward infinity with n then neither does $d^v_{tel}(x_0, x_n)$ since horizontal geodesics are v-telescopic chains. We so proved item (b).

To prove item (c), it is sufficient to check that for any $v \geq 0$, the identity-map is a quasi isometric embedding from $(\widetilde{X}, d_{\widetilde{X}})$ to $(\widetilde{X}, d^v_{tel})$. For this sake, just observe that there is some constant $X(v)$ such that the v-telescopic length of a 0-telescopic chain g

of $(\widetilde{X}, d_{\widetilde{X}})$ is bounded above by $X(v)$ times the length of g plus $X(v)$. Conversely, there are constants $Y(v) \geq 1$ and $Z(v) \geq 0$ such that any v-telescopic chain g is at Hausdorff distance smaller than $Z(v)$ from a 0-telescopic chain whose length is bounded above by $Y(v)$ times the v-telescopic length of g plus $Y(v)$. From the associated inequalities, we get that the identity-map is a quasi-isometric embedding from one space to the other, so that item (c) is proved since the identity is surjective.

Item (d) amounts to saying that the image of a geodesic under a (a, b)-quasi isometric embedding is $C(a, b)$-close to any geodesic between the images of the endpoints. This is a well-known assertion, see for instance [7].

To prove item (e) observe that by the definition a v-vertical segment is a geodesic between its endpoints for the v-telescopic distance. Item (e) then follows from item (c).

It suffices to prove item (f) when the edge-paths p_i are edges e_i. The telescopic distance between the endpoints of the concatenation of the s_j is at most $n(v+1)+nv$. Since the length of $e_1 \cdots e_n$ is n, we get that the concatenation of the s_j is a $2v$-vertical segment. □

Remark 3.4. Throughout the remainder of this text, the constants appearing in each lemma, corollary or proposition will be denoted by $C, D, \cdots$ and thereafter they will be referred to by the same letter with the number of the lemma, corollary or proposition in subscript. For instance, if Lemma 3.4 introduces the constants C and D we will refer to these constants as $C_{3.4}$ and $D_{3.4}$.

3.2. **Exponential separation.** A subset S of a Gromov hyperbolic space X is *quasi convex* if there exists a constant C such that any geodesic (quasi geodesic, with the constant C then depending on the constants of "quasi geodesicity", in the case where X is a quasi geodesic space) between any two points in S is contained in the C-neighborhood of S.

Definition 3.5. Let $\widetilde{X}$ be a tree of hyperbolic spaces and let S be a horizontal subset which is quasi convex in its stratum, for the horizontal metric. If x is any point in $\widetilde{X}$ then a *horizontal quasi projection of x to S*, denoted by $P_S^{hor}(x)$, is any point y in S such that $d_{hor}(x, y) \leq d_{hor}(x, S) + 1$.

If x and S do not belong to a same stratum, such a horizontal quasi projection does not exist, the horizontal distance $d_{hor}(x, y)$ being infinite for any $y \in S$.

Lemma 3.6. *Let $\delta \geq 0$ and let $(\mathcal{T}, \{X_e\}, \{X_v\}, \{\jmath_e\})$ be a tree of δ-hyperbolic spaces.*

- *(a) There exists $C \geq 0$ such that if $v \geq C$, if $h \subset X_{t(e)}$ is contained in the horizontal 2δ-neighborhood of some horizontal geodesic between two points in $\jmath_e(X_e)$ then there are v-vertical segments over e starting at any point of h.*
- *(b) For any $v \geq C$, if e is an edge of $\mathcal{T}$ and h is a horizontal geodesic in $X_{t(e)}$ such that no v-vertical segment starting at h can be defined over e, then*

$$\mathrm{diam}_{X_{t(e)}}(P_h^{hor}(\jmath_e(X_e))) \leq 4\delta + 1.$$

- *(c) For any $v \geq C$, if for any edge e of $\mathcal{T}$ the map $\jmath_e$ is a $(\mathfrak{a}, \mathfrak{b})$-quasi isometry (and not only a $(\mathfrak{a}, \mathfrak{b})$-quasi isometric embedding - see Definition 2.5) then there is a v-vertical segment over any $\mathcal{T}$-geodesic through any point of $\widetilde{X}$ (the geometric realization of the tree of δ-hyperbolic spaces).*

Proof of Lemma 3.6. By definition of a tree of δ-hyperbolic spaces, each stratum is δ-hyperbolic for the horizontal metric. Moreover the map $\jmath_e$ is a $(\mathfrak{a}, \mathfrak{b})$-quasi isometric embedding. This gives a constant c such that for any two points $x, y \in \jmath_e(X_e)$, any horizontal geodesic $[x, y]$ lies in the horizontal c-neighborhood of the image of $\jmath_e$. Of course, any point $x \in X_{t(e)}$ in this image is the endpoint of a 0-vertical segment over e: it suffices to map $t(e)$ to x and choose the isometric embedding of e into $\widetilde{X}$ to be the interval $\{x\} \times (0,1)$. Since h is contained in the horizontal 2δ-neighborhood of some horizontal geodesic between two points in the image of $\jmath_e$ then any point in h is at horizontal distance at most $2\delta + c$ of some point in this image and therefore of the endpoint of a 0-vertical segment over e. Thus any point in h is the initial point of some v-vertical segment over e as soon as v is chosen sufficiently large with respect to $2\delta + c$. We so get item (a).

Let us prove item (b). Let h be a horizontal geodesic. If there exist x, y in the image of $\jmath_e$ such that $[x,y] \cap \mathcal{N}^{2\delta}_{hor}(h) \neq \emptyset$, then it was proved above that, if $v \geq C$, there is a point in h which is the initial point of a v-vertical segment over e. Since it is assumed that no such v-vertical segment exists, we get $[x,y] \cap \mathcal{N}^{2\delta}_{hor}(h) = \emptyset$ for any x, y in the image. By the 2δ-thiness of the geodesic rectangles, it follows that $\mathrm{diam}_{X_{t(e)}}(P^{hor}_h(\jmath_e(X_e))) \leq 4\delta + 1$: otherwise there is $c \in P^{hor}_h(\jmath_e(X_e))$, $x, y \in \jmath_e(X_e)$, $x' \in P^{hor}_h(x)$ and $y' \in P^{hor}_h(y)$ such that c is at horizontal distance greater than 2δ from $[x,x']$ and $[y,y']$. The 2δ-thiness of the rectangle with corners a, b, a', b' implies that c is at horizontal distance smaller than 2δ from $[x,y]$ which is a contradiction with which precedes. We so got item (b).

Finally, since each map $\jmath_e$ is a $(\mathfrak{a}, \mathfrak{b})$-quasi isometry, by definition, any point in any vertex-space is at horizontal distance at most $\mathfrak{b}$ from the endpoint of a 0-vertical segment over any edge of $\mathcal{T}$ incident to this vertex. Thus, there is a section of π over $\mathcal{T}$ such that the horizontal deviation in the stratum over each vertex is at most $\mathfrak{b}$. By items (b) and (f) of Lemma 3.3 such a section of π is a v-quasi isometric embedding of $\mathcal{T}$ as soon as v is sufficiently large with respect to $\mathfrak{b}$, hence the conclusion. $\square$

We end this section with a simple lemma about the constants of hyperbolicity.

Lemma 3.7. *Let $(\widetilde{X}, \mathcal{T})$ be a tree of hyperbolic spaces satisfying the exponential-separation property.*

(a) The constants of hyperbolicity and quasi isometry can be chosen arbitrarily large.

(b) For any constants of hyperbolicity λ, M, t_0 such that M is sufficiently large, there exists $C \geq 0$ such that the following holds:

For any $\alpha \in \mathcal{T}$, for any β with $d_{\mathcal{T}}(\alpha, \beta) = t_0$, for any two v-vertical segments s_0, s_1 over $[\beta, \alpha]$ such that $d_{hor}(s_0 \cap X_\alpha, s_1 \cap X_\alpha) \geq M$, if

$$\frac{1}{\lambda} d_{hor}(s_0 \cap X_\alpha, s_1 \cap X_\alpha) < d_{hor}(s_0 \cap X_\beta, s_1 \cap X_\beta),$$

then, for any $n \geq 1$, for any $\mathcal{T}$-geodesic ω starting at α with $[\alpha, \beta] \subset \omega$ and $|\omega|_{\mathcal{T}} \geq C + nt_0$:

$$d_{hor}(\omega(s_0 \cap X_\alpha), \omega(s_1 \cap X_\alpha)) \geq \lambda^n d_{hor}(s_0 \cap X_\alpha, s_1 \cap X_\alpha),$$

where the notation ωx denotes the set of points $y \in \widetilde{X}$ such that some v-vertical segment s with $\pi(s) = \omega$ connects x to y.

Proof. Item (a) is obvious. We just give a sketch of a proof of item (b) and leave the remaining details to the reader. Observe that the assumption of item (b) tells us that the horizontal distance between the points $s_0 \cap X_\alpha$ and $s_1 \cap X_\alpha$ is not exponentially contracted with factor $1/\lambda$ in X_β, or equivalently the points $s_0 \cap X_\beta$ and $s_1 \cap X_\beta$ are not exponentially separated in X_α. Thus they are exponentially separated in the other directions: this is the idea developed below.

By the exponential-separation property, the assumption $\frac{1}{\lambda} d_{hor}(s_0 \cap X_\alpha, s_1 \cap X_\alpha) < d_{hor}(s_0 \cap X_\beta, s_1 \cap X_\beta)$ implies that the points $s_0 \cap X_\beta, s_1 \cap X_\beta$ are exponentially separated in every direction at β which does not intersect $[\alpha, \beta)$. Now, there is a constant $D(v)$ such that, when considering other v-vertical segments s'_0, s'_1 starting respectively at $s_0 \cap X_\alpha$ and $s_1 \cap X_\alpha$, $d_{hor}(s'_i, s_i) \leq D(v)$. Thus, taking M and C sufficiently large, we get that the multiplication by $\lambda^{E[C/t_0]}$ (where $E[.]$ is the integer part) between α and ζ with $d_{\mathcal{T}}(\zeta, \alpha) = C$ and $\beta \in [\alpha, \zeta]$ is sufficiently large to compensate the horizontal deviation, which only consists of summing constants $D(v)$, caused by the fact that v-vertical segments over a same geodesic of $\mathcal{T}$ do not necessarily end at the same points: we so get, when C is sufficiently large, a multiplication by λ of $d_{hor}(s_0 \cap X_\alpha, s_1 \cap X_\alpha)$. The exponential-separation property then implies that for each nt_0 thereafter one has a multiplication by λ^n. $\square$

4. Approximation of quasi geodesics: a "simple" case

The corridors (and later the generalized corridors) defined below are not the hallways of [2]. The reason is that we are interested in exhibiting quasi convex subsets of our trees of hyperbolic spaces and the hallways of [2], in general, are not quasi convex.

Definition 4.1. Let $(\widetilde{X}, \mathcal{T}, \pi)$ be a tree of hyperbolic spaces, and let $v \geq 0$.

A *v-vertical tree* is a $(v+1, v)$-quasi isometric embedding $\sigma \colon T \to \widetilde{X}$ of a subtree T of $\mathcal{T}$ into $(\widetilde{X}, d_{\widetilde{X}})$ which is a section of π.

A v-vertical tree $\sigma \colon T \to \widetilde{X}$ is *maximal* if and only if there exists no v-vertical tree $\sigma' \colon T' \to \widetilde{X}$ such that $T \subset T'$, $T \neq T'$ and $\sigma'_{|T} = \sigma$.

A *v-corridor* $\mathcal{C}$ is a subset of $\widetilde{X}$ for which there exist two maximal v-vertical trees $\sigma_i \colon T_i \to \widetilde{X}$ ($i = 1, 2$) termed the *vertical boundaries* of $\mathcal{C}$, with the following properties:

(a) If $T = T_1 \cap T_2$ then for each $\alpha \in T$, $\mathcal{C} \cap X_\alpha$ is a horizontal geodesic with endpoints $\sigma_1(T_1) \cap X_\alpha$ and $\sigma_2(T_2) \cap X_\alpha$.
  The subtree T of $\mathcal{T}$ is *the core* of $\mathcal{C}$ and the union of the horizontal geodesics in the strata over the valency 1 vertices of T, if any, is the *horizontal boundary* of $\mathcal{C}$.

(b) For any $\alpha \in T_i \setminus T$, $\mathcal{C} \cap X_\alpha = \sigma_i(\alpha)$.

(c) For any $\alpha \notin T_1 \cup T_2$, $\mathcal{C} \cap X_\alpha = \emptyset$.

Remark 4.2. Let $(\widetilde{X}, \mathcal{T}, \pi)$ be a tree of hyperbolic spaces the attaching-maps of which are all quasi isometries (and not only quasi isometric embeddings). Then, from item (c) of Lemma 3.6, as soon as $v \geq C_{3.6}$, given any two points x, y in $\widetilde{X}$ there is a v-corridor $\mathcal{C}$ with $Core(\mathcal{C}) = \mathcal{T}$, whose vertical boundaries $\sigma_i \colon \mathcal{T} \to \widetilde{X}$ pass through x and y.

Definition 4.3. Let $\mathcal{C}$ be a union of horizontal geodesics in a tree of hyperbolic spaces $(\widetilde{X}, \mathcal{T})$. Assume that for each stratum X_α the intersection $\mathcal{C} \cap X_\alpha$ is either empty or a horizontal geodesic.

If x is a point in a stratum X_α such that $\mathcal{C} \cap X_\alpha$ is non-empty, then $P^{hor}_{\mathcal{C}}(x)$ stands for the horizontal quasi projection $P^{hor}_{\mathcal{C} \cap X_\alpha}(x)$ of x to $\mathcal{C}$ (see Definition 3.5).

In the definition above, for instance $\mathcal{C}$ might be a corridor. Before stating Lemma 4.4 below, we would like to insist on the fact that the horizontal quasi projection $P^{hor}_{\mathcal{C}}$ is a projection *in the strata* which only refers to the horizontal metric defined on each stratum.

Lemma 4.4. *Let $(\widetilde{X}, \mathcal{T}, \pi)$ be a tree of hyperbolic spaces.*
For any $v \geq C_{3.6}$, there exists $C \equiv C(v) \geq v$ such that:

(a) If $\mathcal{C}$ is a v-corridor in $\widetilde{X}$ then for any v-vertical segment s with $\pi(s)$ contained in the core of $\mathcal{C}$, $P^{hor}_{\mathcal{C}}(s)$ is a C-vertical segment.

(b) *For any v-corridor $\mathcal{C}$ in $\widetilde{X}$, the $C(v)$-telescopic distance $d_{tel}^{C(v)} \colon \mathcal{C} \times \mathcal{C} \to \mathbb{R}^+$, which is the infimum of the lengths of the $C(v)$-telescopic chains in $\mathcal{C}$ between the considered points, is well-defined and $(\mathcal{C}, d_{tel}^{C(v)})$ is a quasi geodesic metric space.*

Proof. If $\sigma \colon \omega \to \widetilde{X}$ is the section of π such that $s = \sigma(\omega)$ then $P_{\mathcal{C}}^{hor}(s)$ is the image of ω under the map $P_{\mathcal{C}}^{hor} \circ \sigma$. This map is a section of π since the horizontal quasi projection $P_{\mathcal{C}}^{hor}$ is a projection in each stratum. We want to prove the existence of $C(v)$ independent of ω such that $P_{\mathcal{C}}^{hor} \circ \sigma$ is a $(C(v)+1, C(v))$-quasi isometric embedding of ω into $\widetilde{X}$. Assume that ω is (contained in) a single edge. Let $z = (P_{\mathcal{C}}^{hor} \circ \sigma)(i(\omega))$. Since $v \geq C_{3.6}$ and since $\mathcal{C}$ is a v-corridor, if it is defined over ω, by item (a) of Lemma 3.6 v-vertical segments can be defined over ω starting at z. By item (d) of Lemma 3.3, the endpoint z' of a v-vertical segment starting at z is in the horizontal $C_{3.3}(1, v)$-neighborhood of $\mathcal{C}$. Consider a horizontal geodesic rectangle with vertices $z', \sigma(t(\omega)), P_{\mathcal{C}}^{hor}(z')$ and $(P_{\mathcal{C}}^{hor} \circ \sigma)(t(\omega))$. By the δ-hyperbolicity of the strata, it is 2δ-thin. From item (a) of Lemma 3.3, the length of the subgeodesic of $[z', \sigma(t(\omega))]$ which is in the horizontal 2δ-neighborhood of $[P_{\mathcal{C}}^{hor}(z'), (P_{\mathcal{C}}^{hor} \circ \sigma)(t(\omega))]$ is bounded above by some constant only depending on δ and on the constants of quasi isometry embeddings for v-vertical segments. Thus the point z' is at bounded horizontal distance from $(P_{\mathcal{C}}^{hor} \circ \sigma)(t(\omega))$, the bound being independent from the chosen v-vertical segment s and edge ω of $\mathcal{T}$. We so easily get that $P_{\mathcal{C}}^{hor} \circ \sigma$ is a $(d(v), d(v) + 1)$-quasi isometric embedding for some constant $d(v)$ if the image of $\pi \circ \sigma$ is (contained in) an edge of $\mathcal{T}$. By item (f) of Lemma 3.3, a concatenation of $d(v)$-vertical segments s_i such that the terminal point of s_i is the initial point of s_{i+1}, and which projects under π to a geodesic of $\mathcal{T}$, defines a $2d(v)$-vertical segment. This proves item (a). Setting $C(v) = 2d(v)$, it readily follows that any two points $x, y \in \mathcal{C}$ are connected by a $C(v)$-telescopic chain in $\mathcal{C}$ so that the distance $d_{tel}^{C(v)}(x, y)$ is never infinite (and of course never zero if $x \neq y$). The $C(v)$-telescopic metric on $\mathcal{C}$ is then well-defined. It makes $\mathcal{C}$ a quasi geodesic space in the same way as the telescopic distance makes $\widetilde{X}$ a quasi geodesic space: we got item (b). $\square$

In Definition 4.5 below, beware that the "diagonal distance" we introduce does not satisfy the triangular inequality.

Definition 4.5. Let $(\widetilde{X}, \mathcal{T}, \pi)$ be a tree of qi-embedded geodesic spaces, let $v \geq 0$.

(a) The *diagonal distance* between two maximal v-vertical trees $\sigma_i \colon T_i \to \widetilde{X}$ is the infimum of the horizontal lengths of the horizontal geodesics between $\sigma_1(T_1)$ and $\sigma_2(T_2)$. The diagonal distance is infinite if no such horizontal geodesic exists.

(b) A *diagonal between two maximal v-vertical trees* $\sigma_i \colon T_i \to \widetilde{X}$ is any horizontal geodesic D between $\sigma_1(T_1)$ and $\sigma_2(T_2)$ which is contained in some vertex-space, and whose horizontal length is less or equal to the diagonal distance plus 1.

(c) A *diagonal* is a horizontal geodesic D for which there exist two maximal v-vertical trees $\sigma_i \colon T_i \to \widetilde{X}$ passing through its endpoints such that D is a diagonal between $\sigma_1(T_1)$ and $\sigma_2(T_2)$.

The diagonal distance between two distinct vertical trees may vanish and so a diagonal may be reduced to a single point.

Before the statement of Theorem 4.6, we would like to point out that this is a theorem about trees of hyperbolic spaces whose attaching-maps are quasi isometries, and not only quasi isometric embeddings. This simplification has an important consequence, see Remark 4.2. The main feature of this theorem is to approximate quasi geodesics of $\widetilde{X}$ by "canonical" quasi geodesics.

Theorem 4.6. *Let $\widetilde{X}$ be a tree of hyperbolic spaces which satisfies the exponential-separation property. Assume that each attaching-map from an edge-space into a vertex-space is a quasi isometry. Then for any $v \geq C_{3.6}$, for any L greater than some critical constant, for any $a \geq 1$ and $b \geq 0$, there are $E \equiv E(v) \geq v$, $D \equiv D(L, v) \geq L$, $C \equiv C(L, v, a, b) \geq 0$, such that the following holds:*

For any v-corridor $\mathcal{C}$ whose vertical boundaries B_0 and B_1 are at diagonal distance at least D, there is an E-telescopic chain $\mathcal{P} = (h_0, s_0, h_1, \cdots, s_{k-1}, h_k)$ in $\mathcal{C}$ satisfying the following properties:

(a) $\mathcal{P}$ is a (D, D)-quasi geodesic of $(\widetilde{X}, d_{\widetilde{X}})$ which connects B_0 to B_1.

(b) For any $0 \leq i \leq k-1$, h_i is a length L diagonal and $|h_k|_{hor} \leq D$.

(c) For any (a, b)-quasi geodesic g in $\widetilde{X}$ with endpoints in B_0 and B_1, if $t_i \subset B_i$ is the v-vertical segment in B_i from $\mathcal{P}$ to g then $(*, t_0^{-1}, \mathcal{P}, t_1, *)$, where $*$ denotes the trivial (i.e. degenerate to a point) horizontal path, defines a (D, D)-quasi geodesic E-telescopic chain whose Hausdorff distance from g in $(\widetilde{X}, d_{\widetilde{X}})$ is bounded above by C.

If $\mathcal{C}$ is a v-corridor whose vertical boundary trees B_0, B_1 are at diagonal distance smaller than D then any (a, b)-quasi geodesic g in $\widetilde{X}$ with endpoints in B_0 and B_1 is contained in the telescopic C-neighborhood of $B_0 \cup B_1$. More precisely, g is at Hausdorff distance smaller than C from a telescopic chain of the form $(*, t_0^{-1}, h_0, t_1, *)$ where h_0 is a (possibly degenerate to a point) horizontal geodesic in $\mathcal{C}$ with horizontal length smaller than D, and t_i is the v-vertical segment in B_i from h_0 to g.

For proving this theorem, we need two important propositions which we state now but whose proofs are postponed until Sections 8 and 9. For the understanding of Proposition 4.7, let us recall that we proved in Lemma 4.4 that the w-telescopic metric is well-defined over any v-corridor $\mathcal{C}$ in a tree of hyperbolic spaces $\widetilde{X}$ as soon as w is sufficiently large. The corridor $\mathcal{C}$ then becomes a quasi geodesic metric space when equipped with this telescopic metric and this quasi geodesic metric space is denoted by $(\mathcal{C}, d^w_{tel})$.

Proposition 4.7. *Let $\widetilde{X}$ be a tree of hyperbolic spaces which satisfies the exponential-separation property.*

For any $v \geq C_{3.6}$, there is $D \geq v$ such that for any $w \geq D$, $L > 0, a \geq 1$ and $b \geq 0$ there exists $C \geq 0$ such that if $\mathcal{C}$ is a v-corridor in $\widetilde{X}$, if L is the horizontal length of some horizontal geodesic $[x, y]$ in $\mathcal{C}$, if g is a (a, b)-quasi geodesic of $(\mathcal{C}, d^w_{tel})$ from T_x, a w-vertical tree through x in $\mathcal{C}$, to T_y, a w-vertical tree through y in $\mathcal{C}$, then g is contained in the C-neighborhood of the union of the w-vertical segments which connect its endpoints to x and y and are contained respectively in T_x and T_y. If L is greater than some critical constant then for any $L' > L'' \geq L$, we have $C(L') > C(L'') \geq C(L)$.

See Section 8 for a proof.

Proposition 4.8. *Let $\widetilde{X}$ be a tree of hyperbolic spaces which satisfies the exponential-separation property and the attaching-maps of which are quasi isometries.*

For any $v \geq C_{3.6}$, $a \geq 1$ and $b \geq 0$ there exists $C \geq 0$ such that, if g is a (a, b)-quasi geodesic in $\widetilde{X}$ and if $\mathcal{C}$ is a v-corridor whose vertical boundaries pass through the endpoints of g then

$$g \subset \mathcal{N}^C_{\widetilde{X}}(\mathcal{C}),$$

where $\mathcal{N}^C_{\widetilde{X}}(\mathcal{C})$ denotes the C-neighborhood of $\mathcal{C}$ for the telescopic metric.

See Section 9 for a proof.

We will also need the following two much easier statements.

Lemma 4.9. *Let $\widetilde{X}$ be a tree of hyperbolic spaces.*

There exists $C \geq 0$ such that for any $v \geq 0$, for any v-corridor $\mathcal{C}$ in $\widetilde{X}$, for any two points x, y in a same stratum intersected by $\mathcal{C}$, $d_{hor}(P^{hor}_{\mathcal{C}}(x), P^{hor}_{\mathcal{C}}(y)) \leq d_{hor}(x, y) + C$. The same inequality holds for the horizontal quasi projections of x and y to the image of the embedding of an edge-space into a vertex-space.

Proof. Since there is $\delta \geq 0$ such that strata are δ-hyperbolic spaces for the horizontal metric and the subspaces to which one projects are quasi convex subsets of their stratum for this horizontal metric, this is a consequence of [7], Corollary 2.2. □

Lemma 4.10. *Let $\widetilde{X}$ be a tree of hyperbolic spaces.*

For any $v \geq C_{3.6}$, $a \geq 1$ and $b, r \geq 0$ there are $C \equiv C(v, a, b, r) \geq 1$ and $D \equiv D(v)$ such that for any v-telescopic (a, b)-quasi geodesic g of $\widetilde{X}$ and for any v-corridor $\mathcal{C}$, if $g \subset \mathcal{N}^r_{hor}(\mathcal{C})$ then any horizontal quasi projection $P^{hor}_{\mathcal{C}}(g)$ is a D-telescopic (C, C)-quasi geodesic of $(\mathcal{C}, d^D_{tel})$.

Proof. By Lemma 4.4 $P^{hor}_{\mathcal{C}}(g)$ is a $C_{4.4}(v)$-telescopic chain. Let us consider any two points x, y in $G = P^{hor}_{\mathcal{C}}(g)$. They are r-close to two points x', y' in g. We denote by $g_{x'y'}$ the subpath of g between x' and y' and by G_{xy} the subset of G between x and y. Since we now consider the $C_{4.4}(v)$-telescopic distance, $|G_{xy}|^{C_{4.4}(v)}_{vert} = |g_{x'y'}|^v_{vert}$. From Lemma 4.9 and since any two horizontal paths in G are separated by a vertical segment of vertical length at least 1, we then get $|G_{xy}|^{C_{4.4}(v)}_{tel} \leq 2C_{4.9}|g_{x'y'}|^v_{tel}$. Since g is a v-telescopic (a, b)-quasi geodesic, $|g_{x'y'}|^v_{tel} \leq a d^v_{tel}(x', y') + b$. But $d^v_{tel}(x', y') \leq 2r + d^v_{tel}(x, y)$. Therefore:

$$|G_{xy}|^{C_{4.4}(v)}_{tel} \leq 2C_{4.9}(a(2r + d^v_{tel}(x, y)) + b).$$

Since all telescopic distances are quasi isometric (item (c) of Lemma 3.3), we so get the right inequality for the quasi geodesicity of $P^{hor}_{\mathcal{C}}(g)$. We leave it to the reader to work out the straightforward proof of the existence of constants $A \geq 1$ and $B \geq 0$ such that $1/A\ d^v_{tel}(x, y) - B \leq |G_{xy}|^{C_{4.4}(v)}_{tel}$. □

Proof of Theorem 4.6. Let $\widetilde{X}$ be a tree of hyperbolic spaces which satisfies the exponential-separation property and such that each attaching-map from an edge-space into a vertex-space is a quasi isometry. Let $v \geq C_{3.6}$. Let $\mathcal{C}$ be any v-corridor. Since the attaching maps of the tree of hyperbolic spaces are all quasi isometries, $Core(\mathcal{C}) = \mathcal{T}$. It follows from Lemma 4.4 that $(\mathcal{C}, d^{C_{4.4}}_{tel})$ is a quasi geodesic metric space. Since $\widetilde{X}$ satisfies the exponential-separation property, the $C_{4.4}$-vertical segments are exponentially separated. From item (b) of Lemma 3.7, this implies in particular that the endpoints of any diagonal with horizontal length greater than some constant M are exponentially separated in all the directions of $\mathcal{T}$ outside a region with vertical width bounded by $2C_{3.7}$.

Since $v \geq C_{3.6}$, there are v-vertical trees over $\mathcal{T}$ through any point of $\widetilde{X}$. By Lemma 4.4, the horizontal projections of these trees to $\mathcal{C}$ give $C_{4.4}$-vertical trees over $\mathcal{T}$ through the points of $\mathcal{C}$.

Let $L \geq M$. Consider a length L diagonal h_0 from the vertical boundary B_0 of $\mathcal{C}$ to some maximal $C_{4.4}$-vertical tree T_0 in $\mathcal{C}$, we assume for a while that such a T_0 exists. Then another length L diagonal h_1 from T_0 to another maximal $C_{4.4}$-vertical tree T_1 such that the diagonal distance in $\mathcal{C}$ between T_1 and B_1 is strictly smaller than the diagonal distance between T_0 and T_1, and so on until arriving at a maximal $C_{4.4}$-vertical tree T_r for which there exists no other $C_{4.4}$-vertical tree in $\mathcal{C}$ connected to T_r by a length L diagonal. Then, as a consequence of the exponential-separation property, there is an upper-bound

$d_0(L,v)$ to the diagonal distance from T_r to the other v-vertical boundary tree B_1 of $\mathcal{C}$. An ordered sequence composed of:

- the diagonals $h_0, h_1, \cdots, h_r$,
- a horizontal geodesic h_{r+1} with $|h_{r+1}|_{hor} \leq d_0(L,v)+1$ between T_r and B_1 such that $d_{vert}(h_{r+1}, h_r) \leq d_{vert}(h, h_r)$ for any horizontal geodesic h with the same properties,
- the $C_{4.4}$-vertical segments in $T_0, T_1, \cdots, T_r$ between the endpoints of the h_i's

gives to us the telescopic chain denoted by $\mathcal{P}$ in Theorem 4.6. Moreover, if x (resp. y) is a point in the vertical boundaries of $\mathcal{C}$, there is an unique v-vertical segments t_0 in B_0 (resp. t_1 in B_1) between the horizontal geodesic h_0 (resp. h_{r+1}) and the point x (resp. y). Adding these v-vertical segments t_0 and t_1 gives the telescopic chain announced by item (c) as we are now going to check: we denote this last telescopic chain by $\mathcal{P}_x^y$.

Let g be any (a,b)-quasi geodesic of $\widetilde{X}$ between x and y. By Proposition 4.8, $g \subset \mathcal{N}_{tel}^{C_{4.8}}(\mathcal{C})$. By item (c) of Lemma 3.3, there is some constant $Z(v)$ such that g is $Z(v)$-Hausdorff close to some v-telescopic (a',b')-quasi geodesic chain: for the sake of simplicity, we still denote by g this v-telescopic chain and by a and b its constants of quasi geodesicity. From Lemma 4.10, $\mathfrak{G} \equiv P_{\mathcal{C}}^{hor}(g)$ is a $D_{4.10}$-telescopic $(C_{4.10}, C_{4.10})$-quasi geodesic of $(\mathcal{C}, d_{tel}^{D_{4.10}})$.

The quasi geodesic $\mathfrak{G}$ intersects the vertical trees $T_0, T_1, \cdots$ of $\mathcal{C}$: let $\mathfrak{G}_0$ be the shortest initial segment of g that connects x to T_0. From Proposition 4.7, $\mathfrak{G}_0$ is contained in the $C_{4.7}$-neighborhood of the union of the vertical segments s_0, s_1 from the endpoints of $\mathfrak{G}_0$ to those of h_0. From our observation above about the exponential separation of the endpoints of h_0, there is some $\kappa > 0$ such that, outside the region in $\mathcal{C}$ centered at h_0 with vertical width κ, the horizontal geodesics between the vertical trees of the endpoints of h_0 have horizontal length greater than $3C_{4.7}$. We so get a constant $K \equiv K(v,L,a,b) > 0$, not depending of the quasi geodesic nor on the corridor considered, such that $d_{tel}^H(\mathfrak{G}_0, s_0 \cup h_0 \cup s_1) \leq K(v,L,a,b)$ (we recall that d_{tel}^H denotes the Hausdorff distance associated to the $D_{4.10}$-telescopic distance).

The same arguments apply for the subset $\mathfrak{G}_i$ between T_{i-1} and T_i until $i=r$. Since $|h_{r+1}|_{hor} \leq d_0(L,v)+1$ and h_{r+1} has been chosen to minimize the vertical distance between h_r and all horizontal geodesics h satisfying $|h|_{hor} \leq d_0(L,v)+1$, we easily get a constant $K' \equiv K'(v,L,a,b)$ such that the concatenation of h_{r+1} with

- the v-vertical segment in B_1 between h_{r+1} and y (the terminal point of g),
- the $C_{4.4}$-segment in T_r between h_r and h_{r+1},

is at Hausdorff distance smaller than $K'(v,L,a,b)$ from the subset of $\mathfrak{G}$ following the concatenation of the $\mathfrak{G}_i$'s.

It follows that $\mathcal{P}_x^y$ is a $D_{4.10}$-telescopic chain between x and y with $d_{tel}^H(g, \mathcal{P}_x^y) \leq \max(K, K')$. We so proved items (b) and (c) of Theorem 4.6.

It remains to check item (a). It suffices to choose $a=1$ and $b=0$ and then apply what was proved just above: the chain $\mathcal{P}$ is at Hausdorff distance smaller than $\max(K(v,L,1,0), K'(v,L,1,0))$ from a geodesic. Moreover, by construction, the intersections of $\mathcal{P}$ with the strata are horizontal geodesics so that the non-properness of the strata cannot be used to shorten $\mathcal{P}$. From these observations, we easily get by classical arguments and computations that $\mathcal{P}$ is a $(d_1(L,v), d_1(L,v))$-quasi geodesic as announced.

We now deal with the case where there is no $C_{4.4}$-vertical tree in $\mathcal{C}$ which is connected to B_0 by a length L diagonal. Then, as was previously observed when dealing with the non-existence of a similar $C_{4.4}$-vertical tree between T_r and B_1, there is an upper-bound, denoted here $d_2(L,v) \geq L$, on the diagonal distance between B_0 and B_1: the maximum of the constants $d_i(L,v)+1$, $i=1,2,3$, gives the constant $D \geq L$ announced by Theorem 4.6. Let g be any (a,b)-quasi geodesic of $\widetilde{X}$ between $x \in T$ and $y \in T'$. By Proposition 4.8, $g \subset \mathcal{N}_{tel}^{C_{4.8}}(\mathcal{C})$. By item (c) of Lemma 3.3, there is some constant $Z(v)$ such that g is $Z(v)$-Hausdorff close to some v-telescopic (a',b')-quasi geodesic chain: for the sake of simplification, we still denote by g this v-telescopic chain and by a and b its constants of quasi geodesicity. From Lemma 4.10, $\mathfrak{G} \equiv P_{\mathcal{C}}^{hor}(g)$ is a $D_{4.10}$-telescopic $(C_{4.10}, C_{4.10})$-quasi geodesic of $(\mathcal{C}, d_{tel}^{D_{4.10}})$. The assertion of Theorem 4.6 in this case is then a straightforward consequence of Proposition 4.7 applied to $\mathfrak{G}$ if $\mathcal{C}$ contains a horizontal geodesic of length at least D. Otherwise $\mathfrak{G}$ is obviously in a bounded neighborhood of a telescopic chain (h,s) with $|h|_{hor} \leq D$ and the conclusion follows. □

5. Approximation of quasi geodesics: the general case

In order to give a simple statement, we added in Theorem 4.6 the restriction that the attaching-maps of the tree of spaces be quasi isometries, instead of requiring that they be quasi isometric embeddings. In this way, the elementary notion of a corridor (Definition 4.1) was sufficient to describe the quasi geodesics of the space. We now need to introduce the more general notion of *generalized corridor*: the important definition for this purpose is the introduction of the *flat paths* below.

Definition 5.1. Let $(\widetilde{X}, \mathcal{T})$ be a tree of hyperbolic spaces.

A v-telescopic chain $\mathcal{P} = (h_0, s_0, \cdots, h_n)$ is $C(v)$*-flat*, for some constant $C(v) \geq 0$, if there is a union of horizontal geodesics $\mathcal{C}$, at most one in each stratum, which contains $\mathcal{P}$ and satisfies the following properties:

(a) Any two points in $\mathcal{C}$ are connected by a $C(v)$-telescopic path in $\mathcal{C}$.
(b) For any open edge (k,l) of $\mathcal{T}$, $\mathcal{C} \cap \pi^{-1}((k,l))$ is either empty or is a union of maximal horizontal geodesics between two v-vertical segments over (k,l).
(c) If $\mathcal{C} \cap \pi^{-1}((k,l))$ is non-empty, then $\mathcal{C} \cap \pi^{-1}(k)$ and $\mathcal{C} \cap \pi^{-1}(l)$ are non-empty.
(d) If h is a horizontal geodesic in $\mathcal{C}$ over some vertex k of $\mathcal{T}$, $[k,l]$ is a (closed) edge of $\mathcal{T}$ over which v-vertical segments starting at the endpoints of h are defined, and h is maximal in $\mathcal{C} \cap X_k$ with respect to this property, then $\mathcal{C} \cap \pi^{-1}([k,l])$ is a union of horizontal geodesics between two $C(v)$-vertical segments starting at the endpoints of h.
(e) The endpoints of $\mathcal{P}$ are the endpoints of some maximal horizontal geodesics in $\mathcal{C}$.

Such a union of horizontal geodesics which is minimal with respect to the inclusion is a *generalized v-corridor* associated to $\mathcal{P}$.

The *vertical boundary* of a generalized v-corridor $\mathcal{C}$ is the union of all the $C(v)$-vertical trees $\sigma_i \colon T_i \to \widetilde{X}$ such that any point in $\sigma_i(T_i)$ is the endpoint of some maximal horizontal geodesic in $\mathcal{C}$.

By construction, a generalized v-corridor associated to some flat chain $\mathcal{P}$ contains the endpoints of $\mathcal{P}$ in its vertical boundary. By definition, if $\mathcal{C}$ is a generalized v-corridor, then the metric space $(\mathcal{C}, d_{tel}^{C_{5.1}(v)})$ is a quasi geodesic space (this does not mean any kind of "quasi convexity" for these generalized corridors, this property being proved in Section 9).

We recall that the diagonal distance between two maximal vertical trees is infinite if there exists no horizontal geodesic which connects one to the other. In Theorem 5.2 below, by a *diagonal in (the generalized v-corridor) $\mathcal{C}$* we mean a horizontal geodesic which is a diagonal for two maximal $E_{5.2}(v)$-vertical trees in $\mathcal{C}$ passing through its endpoints.

Theorem 5.2. *Let $\widetilde{X}$ be a tree of hyperbolic spaces which satisfies the exponential-separation property. Then for any $v \geq C_{3.6}$, for any L greater than some critical constant, for any $a \geq 1$ and $b \geq 0$ there are $E \equiv E(v) \geq v$, $D \equiv D(L, v) \geq L$ and $C \equiv C(L, v, a, b) \geq 0$ such that the following holds:*

For any two distinct maximal v-vertical trees T_0 and T_1 in $\widetilde{X}$ which are at diagonal distance at least D, there is a E-flat E-telescopic chain $\mathcal{P} = (h_0, s_0, h_1, \cdots, s_{k-1}, h_k)$ and an associated generalized v-corridor $\mathcal{C}$ between T_0 and T_1 satisfying the following properties:

- *(a) $\mathcal{P}$ is a (D, D)-quasi geodesic which connects T_0 and T_1.*
- *(b) For any $0 \leq i \leq k-1$, h_i is a length L diagonal in $\mathcal{C}$ and $|h_k|_{hor} \leq D$.*
- *(c) For any (a, b)-quasi geodesic g in $\widetilde{X}$ with endpoints in T_0 and T_1, if t_i denotes the v-vertical segment in T_i $(i = 0, 1)$ from the endpoint of $\mathcal{P}$ to the endpoint of g then $(*, t_0^{-1}, \mathcal{P}, t_1, *)$, where $*$ denotes the trivial horizontal path, is a (D, D)-quasi geodesic E-telescopic chain whose Hausdorff distance from g in $(\widetilde{X}, d_{\widetilde{X}})$ is bounded above by C.*

If T_0 and T_1 are at diagonal distance smaller than D then any (a, b)-quasi geodesic g in $\widetilde{X}$ with endpoints in T_0 and T_1 is contained in the telescopic C-neighborhood of $T_0 \cup T_1$. More precisely, g is at Hausdorff distance smaller than C from a telescopic chain of the form $(*, t_0^{-1}, h_0, t_1, *)$ where h_0 is a (possibly degenerate to a point) horizontal geodesic in $\mathcal{C}$ with horizontal length smaller than D, and t_i is the v-vertical segment in T_i from h_0 to g.

Proof of Theorem 5.2. The first lemma clarifies the reason for the definition of flat chains and generalized corridors:

Lemma 5.3. *Let $(\widetilde{X}, \mathcal{T}, \pi)$ be a tree of qi-embedded geodesic spaces. Let $v \geq C_{3.6}$. There is $C(v) \geq C_{4.4}(v)$ such that, for any two points x and y in $\widetilde{X}$, there exists a $C(v)$-flat $C(v)$-telescopic chain between x and y, and thus a generalized v-corridor which contains x and y in its vertical boundary. If one is given two maximal v-vertical trees T_0 and T_1 through x and y then this generalized corridor may be chosen to contain T_0 and T_1 in its vertical boundary.*

Proof. By item (f) of Lemma 3.3, concatenating v-vertical segments over the edges in a $\mathcal{T}$-geodesic yields a $2v$-vertical segment: we will call such a concatenation a *v-special concatenation.*

We consider a maximal v-special concatenation s_0, starting at $x \equiv x_0$, over the edges in $[\pi(x), \pi(y)]$. If s_0 does not intersect $X_{\pi(y)}$, let x_1 be its terminal point and $x_2 \in P^{hor}_{X_{i(e)}}(x_1)$ where e is the edge incident to $\pi(x_1)$ in $[\pi(x_1), \pi(y)]$. We repeat the construction with s_1 a maximal v-special concatenation starting at x_2 and so on until s_{k-1} $(k \geq 2)$ intersects the stratum of y. We then denote by x_{2k-1} the point $X_{\pi(y)} \cap s_{k-1}$ and we set $y \equiv x_{2k}$. Let us consider a horizontal geodesic $h_k \equiv [x_{2k-1}, x_{2k}]$. We consider the point x'_{2k-1} in h_k which is the initial point of some non trivial v-special concatenation in the direction of x, and which maximizes the horizontal distance from x_{2k-1} among all the points in h_k sharing this property. The complement of $[x_{2k-1}, x'_{2k-1})$ in $[x_{2k-1}, x_{2k}]$ is denoted by c_k.

If some v-special concatenation s'_{k-1} starting at x'_{2k-1} reaches the stratum of x_{2k-2} then we substitute h_{k-1} by a horizontal geodesic h'_{k-1} between x_{2k-3} and the terminal point of s'_{k-1}. Otherwise we consider a maximal v-special concatenation s'_{k-1} starting at x'_{2k-1} and a horizontal geodesic g^1_{k-1} between s_{k-1} and the terminal point of s'_{k-1}. Then we repeat the construction from g^1_{k-1} (this yields a c^1_k, maybe trivial). Eventually we reach the stratum of x_{2k-2} after building $g^1_{k-1}, g^2_{k-1}, \cdots, g^l_{k-1}$ (and getting $c^1_k, c^2_k, \cdots, c^l_k$). Observe that it might happen that we reach $X_{\pi(x_{2k-2})}$ at x_{2k-2}: in this case, $h'_{k-1} = h_{k-1}$. We repeat the construction starting from h'_{k-1}.

We claim that eventually this construction yields a flat $2v$-telescopic chain between x and y. Indeed it suffices to consider the telescopic chain formed by the c_j's and the c^m_j's (that is the horizontal subgeodesics which lie in some sense in the "boundary" of the subset of $\widetilde{X}$ constructed) and the v-special concatenations which connect them. This telescopic chain is a $2v$-telescopic chain. By construction it is contained in a union $\mathcal{C}'$ of horizontal geodesics, exactly one in each stratum over the points in $[\pi(x), \pi(y)]$. Moreover, by the δ-hyperbolicity of the strata and item (a) of Lemma 3.3, there is a constant $E(v)$ such that h'_j is in the $E(v)$-horizontal neighborhood of the union of h_j and any horizontal geodesic between x_{2j} and the endpoint of h'_j distinct from x_{2j-1}. Therefore there is $D(v)$ such that any two points in $\mathcal{C}'$ are connected by a $D(v)$-telescopic chain in $\mathcal{C}'$. In order to get a whole union of horizontal geodesics $\mathcal{C}$ as required by Definition 5.1, it suffices to complete $\mathcal{C}'$ as follows: consider an edge e of $\mathcal{T}$ incident to some vertex v in $[\pi(x), \pi(y)]$ and not in $[\pi(x), \pi(y)]$; consider a maximal horizontal geodesic in $\mathcal{C}' \cap \pi^{-1}(v)$ whose endpoints are the initial points of v-vertical segments over e; add to $\mathcal{C}'$ the union of two such v-vertical segments with horizontal geodesics between one and the other in the strata over the points in e (one horizontal geodesic in each stratum); still denote by $\mathcal{C}'$ the new subset of $\widetilde{X}$ constructed. Then repeat the construction for each vertex v in $[\pi(x), \pi(y)]$, each edge e incident to v and continue until exhausting $\mathcal{T}$ or until it is impossible to find v-vertical segments over the edges incident to the vertices reached up to now, and not lying in the subtree of $\mathcal{T}$ over which $\mathcal{C}'$ is now defined. We so get a generalized corridor $\mathcal{C}$ associated to our flat path. By Lemma 4.4, taking $C(v)$ equal to the maximum of the constant $D(v)$ above and of $C_{4.4}(v)$, any two points in $\mathcal{C}$ are connected by a $C(v)$-telescopic path in $\mathcal{C}$. Hence our $2v$-telescopic chain is $C(v)$-flat.

If one starts with two given maximal v-vertical trees T_0 and T_1, then we adapt the construction by requiring, when it makes sense, that the vertical segments we construct are contained in these vertical trees. This is possible because our construction yields $2v$-vertical segments, and trees, and v-vertical trees like T_0 and T_1, are in particular $2v$-vertical trees. □

We now need an adaptation to this more general setting of some of the propositions given for proving Theorem 4.6:

Proposition 5.4. *Proposition 4.7 and Proposition 4.8 remain true for generalized v-corridors with $v \geq C_{3.6}$.*

Once Proposition 4.7 is proven in the setting of corridors, there is nothing new to prove in the setting of generalized corridors. We refer the reader to Section 9.6 for the proof of the adaptation of Proposition 4.8 to generalized corridors.

Lemma 5.5. *Let $\widetilde{X}$ be a tree of hyperbolic spaces.*

For any $v \geq C_{3.6}$, for any $a \geq 1$ and $b \geq 0$ there exists $C \geq 0$ such that if g is any (a, b)-quasi geodesic, if $\mathcal{C}$ is any generalized v-corridor the vertical boundaries of which pass through the endpoints of g, then there is a $(a, b + (4\delta + 1))$-quasi geodesic G with $d^H_{\widetilde{X}}(g, G) \leq C$ and $\pi(G) \subset \pi(\mathcal{C})$.

Proof. Assume that g' is a maximal subset of g with endpoints in some vertex-space X_γ and such that $\pi(g') \cap \pi(\mathcal{C}) = \gamma$. This implies that there are no v-vertical segments starting at $\mathcal{C} \cap X_\gamma$ over the (open) edges e incident to γ such that $\pi(g')$ intersects e and

$e \notin \pi(\mathcal{C})$. Then, since $v \geq C_{3.6}$, item (b) of Lemma 3.6 tells us that the endpoints of g' are $(4\delta+1)$-close with respect to the horizontal distance. Since g is a (a,b)-quasi geodesic, g' is $(a(4\delta+1)+b)$-close to X_γ with respect to the telescopic distance. Substituting g' by a horizontal geodesic between its endpoints and repeating this substitution for all the subsets of g like g' yields a quasi geodesic as announced. □

With the above adaptations in mind, the proof of Theorem 5.2 is a duplicate of the proof of Theorem 4.6: in a first step, Lemma 5.3 gives a $C_{5.3}(v)$-flat $C_{5.3}(v)$-telescopic chain between T_0 and T_1, and a generalized corridor $\mathcal{C}$ associated to it. □

6. Weak relative hyperbolicity

The aim of this section is to prove Theorem 2.15. An intermediate result is Theorem 6.1 which generalizes Bestvina-Feighn's combination to non-proper hyperbolic spaces. Bowditch proposed such a generalization in [4].

Theorem 6.1. *Let $\widetilde{X}$ be a tree of hyperbolic spaces which satisfies the exponential-separation property. Then $\widetilde{X}$ is a Gromov-hyperbolic metric space.*

In order to prove this theorem, we need two statements:

Theorem 6.2. *Let $\widetilde{X}$ be a tree of hyperbolic spaces which satisfies the exponential-separation property. For any $a \geq 1$ and $b \geq 0$ there exists $C \geq 0$ such that (a,b)-quasi geodesic bigons are C-thin.*

Proof of Theorem 6.2. By item (c) of Lemma 3.3, it suffices to prove Theorem 6.2 for $C_{3.6}$-telescopic (a,b)-quasi geodesic bigons. Let g_0, g_1 be the two sides of a $C_{3.6}$-telescopic (a,b)-quasi geodesic bigon. By Lemma 5.3 (the points x and y in the statement of this lemma are the endpoints of the bigon $g_0 \cup g_1$) and Theorem 5.2 (Remark 4.2 and Theorem 4.6 suffice in the case where the attaching-maps of $\widetilde{X}$ are quasi isometries), there is $E_{5.2} \geq C_{3.6}$ and an $E_{5.2}$-telescopic chain $\mathcal{P}$ such that for $i = 0,1$ we have $d^H(g_i, \mathcal{P}) \leq C_{5.2}$ in the case where the diagonal distance between two v-vertical trees passing through x and y is at least L. Hence $d^H(g_0, g_1) \leq 2C_{5.2}$ in this case and Theorem 6.2 is proved. In the case where the diagonal distance is smaller than L, since the g_i's have the same endpoints, the last assertion of Theorem 5.2 yield the same conclusion. □

The following lemma was first indicated to the author by I. Kapovich:

Lemma 6.3. [10] *Let (X,d) be a (r,s)-quasi geodesic space. If for any $r' \geq r, s' \geq s$, there exists $\delta(r',s')$, such that (r',s')-quasi geodesic bigons are $\delta(r',s')$-thin, then (X,d) is a $2\delta(r,3s)$-hyperbolic space.*

Proof of Theorem 6.1: Let $\widetilde{X}$ be a tree of hyperbolic spaces which satisfies the exponential-separation property. By Theorem 6.2 the (r,s)-quasi geodesic bigons are $C_{6.2}(r,s)$-thin. Lemma 6.3 gives $\delta = 2C_{6.2}(r,s)$ such that $\widetilde{X}$ is a δ-hyperbolic space, hence Theorem 6.1. □

Proof of Theorem 2.15: Let $(\mathcal{T}, \{(X_e, \mathcal{P}_e)\}, \{(X_v, \mathcal{P}_v)\}, \{\jmath_e\})$ be a tree of weakly relatively hyperbolic spaces. Since $(\mathcal{T}, \{\widehat{X}_e\}, \{\widehat{X}_v\}, \{\widehat{\jmath}_e\})$ satisfies the exponential-separation property, by Theorem 6.1, $(\mathcal{T}, \{\widehat{X}_e\}, \{\widehat{X}_v\}, \{\widehat{\jmath}_e\})$ is hyperbolic. This is exactly equivalent to (the geometric realization of) $(\mathcal{T}, \{(X_e, \mathcal{P}_e)\}, \{(X_v, \mathcal{P}_v)\}, \{\jmath_e\})$ being weakly hyperbolic relatively to the family composed of all the parabolic subspaces of the edge- and vertex-spaces and Remark 2.16 is proved. Since the attaching-maps of the trees of spaces are assumed to be pair-maps, the parabolic subspaces of the edge-spaces are mapped into the parabolic subspaces of the vertex-spaces. Thus the parabolic subspaces of the edge-spaces can be removed from the previous family and (the geometric realization of) $(\mathcal{T}, \{(X_e, \mathcal{P}_e)\}, \{(X_v, \mathcal{P}_v)\}, \{\jmath_e\})$ is weakly hyperbolic relatively to the family composed of all the parabolic subspaces of the vertex-spaces. □

7. Strong relative hyperbolicity

The goal in this section is to prove Theorem 2.20. We need some preliminary lemmas and a proposition the proof of which is postponed to Section 9.7.

Consider any induced tree of parabolic spaces. Since in $\widehat{X}$ a cone has been put over each parabolic space, this induced tree of parabolic spaces is naturally identified to a tree whose vertices are the vertices of the cones over the parabolic spaces. By definition of $\widehat{\jmath}_e$, this tree is further identified to a 0-vertical tree: in order not to add unnecessary additional vocabulary, we call this 0-vertical tree of cone-vertices the *induced tree of parabolic spaces* .

We denote by $C(\widehat{X})$ the metric space obtained from the geometric realization of $(\mathcal{T}, \{\widehat{X}_e\}, \{\widehat{X}_v\}, \{\widehat{\jmath}_e\})$ by putting a cone over (the geometric realization of) each induced tree of parabolic spaces, that is over each one of the associated tree of cone-vertices (see above). We say that a path g *passes through a cone* $\mathcal{L}$ *of* $C(\widehat{X}) \setminus \widehat{X}$ if g contains the vertex of $\mathcal{L}$ in $C(\widehat{X}) \setminus \widehat{X}$.

The sketch of the proof of Theorem 2.20 goes as follows: it is a tautology that any quasi geodesic in $C(\widehat{X})$ admits a decomposition into subpaths contained in $\widehat{X}$ and subpaths in (the closure of) $C(\widehat{X}) \setminus \widehat{X}$. The subpaths in $\widehat{X}$ are quasi geodesics of $\widehat{X}$. The meaning of Lemma 7.2 is that any two quasi geodesics g, g' which are contained in $\widehat{X}$ and which connect two given induced trees of parabolic spaces are close one to each other in $\widehat{X}$, provided their endpoints are close. It is a straightforward consequence of the exponential separation of the induced trees of parabolic spaces given by Lemma 7.1. This latter lemma is itself just a rephrasing of the strong exponential separation property. Lemma 7.3 then strenghtens Lemma 7.2 by asserting that any two quasi geodesics g, g' of $C(\widehat{X})$ with same initial point and with terminal points at most 1-apart in $\widehat{X}$ remain close one to each other provided that the entrance and exit-points of the cones of $C(\widehat{X}) \setminus \widehat{X}$ they pass through are contained in a bounded neighborhood *in* $\widehat{X}$ of some generalized corridor between their endpoints (the result would be false if the bound were only in $C(\widehat{X})$). They also satisfy the Bounded-Parabolic Penetration property if they do not backtrack. Then Proposition 7.4 tells us that this required condition is indeed satisfied (the proof of this proposition is postponed to Section 9.7 because, however intuitively easy - this is a consequence of the exponential separation property - it is technically a bit complicated).

Lemma 7.1. *There exists $C \geq 0$ such that any two induced trees of parabolic spaces which intersect a same stratum are exponentially separated in all the directions outside a region whose vertical width is smaller than C.*

Proof. This is a straightforward consequence of the strong exponential-separation property: the region where the trees are at horizontal distance smaller than the constant of hyperbolicity M has vertical width smaller than some constant depending on M. Outside this region, the horizontal lengths between the two induced trees grow exponentially hence the lemma. □

Lemma 7.2. *For any $a \geq 1$ and $b \geq 0$ there exists $C \geq 0$ such that if g, g' are two (a, b)-quasi geodesics of $\widehat{X}$ between two induced trees of parabolic spaces L_1, L_2 then g, g' admit* *decompositions $g = g_1g_2g_3$ and $g' = g'_1g'_2g'_3$ with the following properties: $g_1 \subset \mathcal{N}^C_{\widehat{X}}(L_1)$, $g'_1 \subset \mathcal{N}^C_{\widehat{X}}(L_1)$, $g_3 \subset \mathcal{N}^C_{\widehat{X}}(L_2)$, $g'_3 \subset \mathcal{N}^C_{\widehat{X}}(L_2)$ and $d^H_{\widehat{X}}(g_2, g'_2) \leq C$. If g and g' have the same endpoints then $d^H_{\widehat{X}}(g, g') \leq C$.*

Proof. This is an easy consequence of Theorem 5.2. For simplicity assume that the attaching-maps of $\widehat{X}$ are quasi isometries so that Theorem 4.6 can be applied. The induced trees of parabolic spaces bound a corridor. Both g and g' are approximated by two chains G and G' which only possibly differ by their first and last vertical segments in L_1 and L_2. These last vertical segments are where g and g' are not necessarily close one to each other if they don't have the same endpoints but are close to the given vertical trees. As written before, the extension to the general case where there is not a corridor, but only a generalized corridor, between the two induced trees, is easily dealt with by using Theorem 5.2 instead of Theorem 4.6. □

We recall that the acronym BPP below stands for Bounded-Parabolic Penetration property, see Definition 2.3.

Lemma 7.3. *For any $v \geq C_{3.6}$, for any $a \geq 1$ and $b, r \geq 0$ there exists $C \geq 0$ such that, if g_1, g_2 are any two (a, b)-quasi geodesics of $C(\widehat{X})$ and $\mathcal{C}$ is any generalized v-corridor whose vertical boundaries pass through the endpoints of g_1, which satisfy the following properties:*

- *(a) the terminal points of g_1 and g_2 are at most 1-apart in $\widehat{X}$, and g_1, g_2 have same initial point in $\widehat{X}$,*
- *(b) the traces $\widehat{g}_i$ of the g_i in $\widehat{X}$ satisfy $\widehat{g}_i \subset \mathcal{N}^r_{\widehat{X}}(\mathcal{C})$ for $i = 1, 2$ that is the traces $\widehat{g}_i$ lie in the r-neighborhood of $\mathcal{C}$* with respect to the distance in $\widehat{X}$,

then $d^H_{C(\widehat{X})}(g_1, g_2) \leq C$, that is each g_i lies in the C-neighborhood of the other with respect to the distance in $C(\widehat{X})$. *Furthermore, if g_1 and g_2 do not backtrack then they satisfy the two conditions required by the BPP with a constant D depending on v, a, b, r.*

We emphasize that this proposition is false if one only requires a bound on the distance in $C(\widehat{X})$ from the g_i's to $\mathcal{C}$. The strategy to prove this lemma is to project the quasi geodesics g_i to $\mathcal{C}$: as in the previous section, the bound in $\widehat{X}$ from the g_i's to $\mathcal{C}$ ensure us that these projections still yield quasi geodesics.

Proof. We may assume that g_1 and g_2 do not backtrack. Otherwise delete the backtracking subpaths with same initial and terminal points. Since g_i is a (a, b)-quasi geodesic, this yields a non-backtracking (a, b)-quasi geodesic g'_i which is contained in g_i and such that each point of g_i is at distance at most b in $C(\widehat{X})$ from some point in g'_i. In what follows, we still denote by g_i the resulting non-backtracking quasi geodesics. For simplicity, assume that $\mathcal{C}$ is a corridor, the adaptation to generalized corridors is straightforward. Moreover, by item (c) of Lemma 3.3, the intersections of g_1 and g_2 in $\widehat{X}$ can be approximated by telescopic quasi geodesics: thus we may assume that $g_i \cap \widehat{X}$ consists of a collection of telescopic quasi geodesics ($i = 1, 2$). We consider the horizontal quasi projections p_i on $\mathcal{C}$ of these collections of telescopic quasi geodesics. From Lemma 4.10, these projections p_i are collections of $(C_{4.10}, C_{4.10})$-quasi geodesics. Recall now that the induced trees of parabolic spaces are assimilated to 0-vertical trees. Let us denote by T_j the horizontal quasi projections of all the subsets of the induced trees of parabolic spaces (that is the associated trees of cone-vertices) which are contained in the r-neighborhood of $\mathcal{C}$ with respect to the distance in $\widehat{X}$. By assumption on the traces $\widehat{g}_i$, these are the only induced trees the g_i, and so the p_i, might pass through. From Lemma 7.1, the exponential-separation property and the hyperbolicity of the strata in $\widehat{X}$, there are K and L, depending on r and $C_{4.4}(0)$ such that:

(a) Each T_j is a K-vertical tree.

(b) Any two T_j are exponentially separated outside a region of vertical width at most L.

We choose a constant $C_\star$, greater than the critical constant of Theorem 4.6 and sufficiently large so that the following property is satisfied: Any two v-vertical segments in $\widehat{X}$ (resp. any two $C_{4.4}(v)$-vertical segments in $\mathcal{C}$) through the endpoints of a diagonal in $\mathcal{N}^r_{\widehat{X}}(\mathcal{C})$ (resp. in $\mathcal{C}$) with horizontal length greater or equal to $C_\star$ are exponentially separated.

Such a constant $C_\star$ exists, because of the exponential separation property (and so depends on the constants of exponential separation). As a consequence of the choice of $C_\star$ of of the exponential-separation property, any horizontal geodesic with horizontal length greater or equal to $C_\star$ is exponentially dilated in all the directions except at most one.

If g_1 and g_2 go through the same trees of cone-vertices, then the p_i intersect the same T_j, where the indices are chosen so that the intersections with T_j and T_{j+1} are consecutive along the p_i. Let $\mathcal{R}_j$ be the region between T_j and T_{j+1} bounded by horizontal geodesics with length $C_\star$. Since any two T_j are exponentially separated outside a region of vertical width L, there is an upper-bound L' on the vertical width of $\mathcal{R}_j$ (depending on the constants of exponential separation and on v). Thus, from Lemma 7.2 and Proposition 4.7, the points where the projections of g_1 and g_2 penetrate and leave a given tree T_j of cone-vertices are close because they are close to $\mathcal{R}_j$.

Let us now assume that g_1 enters in a tree of cone-vertices S but g_2 does not. Of course this also holds for the projections on $\mathcal{C}$, that is p_1 enters in some T_j but p_2 does not. We consider the first time where it occcurs: from the preceding observations, g_1 and g_2 are close one to each other before. We then distinguish three cases:
First case: There is no other tree of cone-vertices at horizontal distance smaller than $C_\star$ from S in the r-neighborhood of $\mathcal{C}$, and the vertical boundary of $\mathcal{C}$ is at horizontal distance greater than $C_\star$ from S. Then p_2, the projection of g_2, has to go to a neighborhood of a diagonal of horizontal length $C_\star$ following T_j whose size is bounded from above by a constant depending on a, b, r, v and the constants of exponential separation: this is Theorem 4.6. It remains before in a bounded horizontal neighborhood of T_j (the quasi projection of the tree of cone-vertices S), the bound depending on a, b, v and r (since the constants of quasigeodesicity of the projections depend on r). Of course p_1 leaves T_j in a same bounded neighborhood of this diagonal. Thus:

- p_1 and p_2 are close one to each other and so g_1 and g_2 remain close in $C(\widehat{X})$.
- The vertical length of the subset of p_1 through the tree T_j is bounded from above by a constant depending on a, b, v and r, and this is also true for the vertical deviation between the entrance and exit-points of g_1 in S.

Unless otherwise specified, the upper-bounds in the second and third cases depend on the same constants as in the first case above.
Second case: There is another tree of cone-vertices S' at horizontal distance smaller than $C_\star$ from S in the r-neighborhood of $\mathcal{C}$. If g_1 does not go through this tree S', the arguments are those either of the first or third case. So we assume g_1 goes through S'. The argument is then similar to the case where both p_i intersected the same trees. By Lemma 7.1, S and S' get exponentially separated outside a region of vertical width L. Thus there exist

horizontal geodesics with horizontal length $C_\star$ between S and S' and all these horizontal geodesics lie in a region of bounded vertical length. By Proposition 4.7, both p_1 and p_2 lie in a bounded neighborhood one of each other, since their initial points are close. Hence g_1 and g_2 remain close in $C(\widehat{X})$. Since g_2 is a quasi geodesic, once again the vertical deviation between the entrance and exit-points of g_1 in S is bounded from above.

Third case: The vertical boundary of $\mathcal{C}$ is at horizontal length smaller than $C_\star$ from S. The entrance-point of g_1 in S is close to a point in g_2. Since g_2 is a (a,b)-quasi geodesic and g_2 does not pass through S, it cannot happen that the subset of p_1 in T_j both has a large vertical length and is at small horizontal distance from the considered vertical boundary. Thus, if it has a large vertical length, then there is a uniquely defined stratum where the horizontal distance between S and the considered vertical boundary is smaller than $C_\star$, and which is closest to the entrance-point of g_1 in S with respect to the vertical distance. From Proposition 4.7, p_2 lies in a bounded neighborhood of T_j until reaching a bounded neighborhood of this stratum. Hence g_1 and g_2 remain close and once again, this gives an upper-bound on the vertical length of p_1 (and thus on the vertical deviation between the entrance and exit-points of g_1 in S) since g_2 is assumed to be a quasi geodesic.

Lemma 7.3 now follows in an easy way:

- To conclude for the thiness of the bigon, just observe that at the end of each case, it is proved that g_1 and g_2 are still close one to each other when g_1 leaves the induced tree S: this allows one to pass to the next similar situation, or to conclude as at the beginning if, after that, g_1 and g_2 pass through the same induced trees.
- To conclude for the BPP property, we need of course the fact that the horizontal metrics on the strata satisfy the BPP property.

□

The following proposition is the generalization to $C(\widehat{X})$ of Proposition 4.8.

Proposition 7.4. *For any $v \geq C_{3.6}$, for any $a \geq 1$ and $b \geq 0$ there exists $C > 0$ such that the following holds:*

If g is any non-backtracking (a,b)-quasi geodesic of $C(\widehat{X})$, if $\mathcal{C}$ is any generalized v-corridor whose vertical boundaries pass through the endpoints of g then any trace $\widehat{g}$ of g in $\widehat{X}$ is contained in the C-neighborhood of $\mathcal{C}$ with respect to the metric of $\widehat{X}$.

See proof in subsection 9.7.

Proof of Theorem 2.20. Let g, g' be two non-backtracking (a,b)-quasi geodesics of $C(\widehat{X})$ with same initial point, and with terminal points at most 1-apart in $\widehat{X}$. We assume for simplicity that the attaching-maps of $\widehat{X}$ are quasi isometries, the adaptation to the general case is easy. There is a corridor $\mathcal{C}$ (in the whole generality only a generalized corridor) the vertical boundaries of which pass through the initial and terminal points of g.

From Proposition 7.4, traces $\widehat{g}$ and $\widehat{g}'$ of g and g' are contained in the $C_{7.4}$-neighborhood of $\mathcal{C}$ with respect to the metric of $\widehat{X}$. The assumptions of Lemma 7.3 are satisfied and this lemma gives us the BPP property.

The arguments for proving the hyperbolicity are similar to those exposed above. The proof goes as follows: If g, g' form a (a,b)-quasi geodesic bigon of $C(\widehat{X})$, one first substitutes it by a non-backtracking (a,b)-quasi geodesic bigon g_0, g'_0 with $d^H_{C(\widehat{X})}(g, g_0) \leq b$, $d^H_{C(\widehat{X})}(g', g'_0) \leq b$. From Proposition 7.4, traces $\widehat{g}_0$ and $\widehat{g}'_0$ of g_0 and g'_0 are contained in the $C_{7.4}$-neighborhood of $\mathcal{C}$ with respect to the metric of $\widehat{X}$. Lemma 7.3 then gives the thinness of the quasi geodesic bigon. As in Section 6, the hyperbolicity follows from Lemma 6.3. □

Remark 7.5. The hyperbolicity of the coned space $C(\widehat{X})$ follows from the quasi convexity of the trees of cone-vertices and from the arguments developed for proving Proposition 1 of [22]. However we re-proved it above when listing the arguments for checking the BPP.

8. Proof of Proposition 4.7

Conventions: The constants of hyperbolicity and of quasi isometry are chosen sufficiently large to satisfy the conclusions of Lemma 3.7, and also sufficiently large so that computations make sense. Moreover the horizontal subsets of the (a, b)-quasi geodesics considered will be assumed to be horizontal geodesics. The hyperbolicity of the strata gives, for any $a \geq 1$ and $b \geq 0$, a positive constant $C(a, b)$ such that any (a, b)-quasi geodesic g may be substituted by another one g' with $d^H_{\widetilde{X}}(g, g') \leq C(a, b)$ and satisfying this latter property.

In the proofs of the various intermediate statements, when referring to a constant provided by an earlier result we will sometimes indicate between parentheses the values of some of the parameters from which it depends.

Our first lemma is about quasi geodesics. It holds not only in a corridor but in the whole tree of hyperbolic spaces.

Lemma 8.1. *Let $(\widetilde{X}, \mathcal{T}, \pi)$ be a tree of hyperbolic spaces which satisfies the exponential-separation property. For any $a \geq 1, b \geq 0$ and for any $v \geq C_{3.6}$ there exist $C \geq 0$ and $D \geq 0$ such that, if g is a (a, b)-quasi geodesic in $\widetilde{X}$, if $[x, y] \subset g \cap X_\alpha$ satisfies $d_{hor}(x, y) \geq C$ then for any $\mathcal{T}$-geodesic ω starting at α with $|\omega|_{\mathcal{T}} \geq D + nt_0$, $n \geq 1$, we have $d_{hor}(\omega x, \omega y) \geq \lambda^n d_{hor}(x, y)$.*

Proof. We denote by $\lambda > 1, M, t_0 \geq 1$ the constants of hyperbolicity and by λ_+, μ the constants of quasi isometry. Let us choose $n_\star(a)$ such that $\frac{a}{\lambda^{n_\star}} < 1$. Solving the inequality $e > a(\frac{1}{\lambda^{n_\star}} e + 2n_\star t_0) + b$ gives us $e(a, b) \geq \frac{2an_\star t_0 + b}{1 - a\frac{1}{\lambda^{n_\star}}}$.
Claim: If $d_{hor}(x, y) \geq e(a, b)$, if x', y' are the endpoints of two v-vertical segments s, s' of vertical length $n_\star t_0$, starting at x and y and with $\pi(s) = \pi(s')$, then for any $\mathcal{T}$-geodesic ω_0 such that $\omega_0 \pi(s)$ is a $\mathcal{T}$-geodesic and $|\omega_0|_{\mathcal{T}} = t_0$, $d_{hor}(\omega_0 x', \omega_0 y') \geq \lambda d_{hor}(x', y')$ holds.
Proof of Claim: Assume the existence of ω with $|\omega|_{\mathcal{T}} = n_\star t_0$ such that for some x', y' with $x \in \omega x'$, $y \in \omega y'$ and $d_{hor}(x', y') \geq M$, $d_{hor}(x, y) \geq \lambda^{n_\star} d_{hor}(x', y')$ holds. Then $\frac{1}{\lambda^{n_\star}} e + 2n_\star t_0$ is the telescopic length of a telescopic chain between x and y. But the inequality given at the beginning of the proof tells us that the existence of such a telescopic chain is a contradiction with the fact that g is a v-telescopic (a, b)-quasi geodesic. Therefore, if $d_{hor}(x, y) \geq e(a, b)$ and $d_{hor}(x, y) \geq \lambda_+^{n_\star}(M + \mu)$ (this last inequality is to assert that $d_{hor}(x', y') \geq M$ - see above), then $d_{hor}(x', y')$ does not increase after t_0 in the direction of the v-vertical segments s, s'. The claim follows from the exponential-separation of the v-vertical segments.

From the inequality given by the Claim, since $d_{hor}(x', y') \geq \lambda_+^{-n_\star}(d_{hor}(x, y) + \mu)$, we easily compute an integer $N_\star$ such that, if ω_0 is as in the Claim but with length $N_\star t_0$ then $d_{hor}([\omega_0 \pi(s)]x, [\omega_0 \pi(s)]y) \geq \lambda d_{hor}(x, y)$. Setting $D = N_\star t_0$ and $C(a, b) = e(a, b)$, the constant computed above, we get the lemma. □

Notations: δ a fixed non negative constant, $(\widetilde{X}, \mathcal{T}, \pi)$ a tree of δ-hyperbolic spaces, $w \geq C_{3.6}$ and $v \geq C_{5.3}(w)$ two constants, $\lambda > 1, M, t_0 \geq 1$ the associated constants of hyperbolicity, λ_+, μ the associated constants of quasi isometry.

Lemma 8.2. *For any $a \geq 1, b \geq 0$, there exists $C \geq 0$ such that if $\mathcal{C}$ is a generalized w-corridor with exponentially separated v-vertical segments, if g is a v-telescopic chain which is a (a,b)-quasi geodesic of $(\mathcal{C}, d^v_{tel})$, if the endpoints x, y of g both lie in a same stratum X_α, if $d_{hor}(x,y) \geq C$ then, for any $\mathcal{T}$-geodesic ω starting at α with $|\omega|_{\mathcal{T}} \geq C + nt_0$, $n \geq 1$, and $\omega \cap \pi(g) = \{\alpha\}$, we have:*

$$d_{hor}(\omega x, \omega y) \geq \lambda^n d_{hor}(x,y).$$

Proof. Let us observe that, if $[p,q]$ is any horizontal geodesic in g then the v-vertical trees of p and q bound a horizontal geodesic $[p',q']$ in $[x,y]$.
Claim: If $d_{hor}(p',q') \geq Cte$ with $Cte \equiv \lambda_+^{t_0}(C_{8.1} + t_0 + \mu)$ then for any ω as given by the current Lemma with $|\omega|_{\mathcal{T}} \geq D_{8.1} + t_0$, $d_{hor}(\omega p', \omega q') \geq \lambda d_{hor}(p',q')$.
Proof of Claim: If p' and q' are not exponentially separated in the direction of p, q after t_0, then, because of the exponential-separation property, they are exponentially separated after t_0 in the direction of ω, which yields the announced inequality. Let us assume that p', q' are separated after t_0 in the direction of $[\pi(p'), \pi(p)]$. Thus $d_{hor}(rp', rq') \geq \lambda^n d_{hor}(p',q')$ for a $\mathcal{T}$-geodesic r with $|r|_{\mathcal{T}} = nt_0$ and $r \cap \omega = \{\alpha\}$. Therefore $d_{hor}(p,q) \geq C_{8.1} + t_0$. Lemma 8.1 then implies that p, q are exponentially separated in the direction of $[\pi(p), \pi(p')]$ after $D_{8.1} + t_0$, and the claim is proved.

There is a finite decomposition of $[x,y] \subset X_\alpha$ in subgeodesics $[p'_j, q'_j]$ with disjoint interiors such that each $[p'_j, q'_j]$ connects two v-vertical trees through the endpoints of a horizontal geodesic in g. We denote by I_D the set of $[p'_j, q'_j]$'s with $d_{hor}(p'_j, q'_j) \geq Cte$ and by I_C the set of the others. Let us choose an integer $n \geq 1$. We consider a stratum X_β with $d_{\mathcal{T}}(\beta, \alpha) = D_{8.1} + nt_0$. Let h be the horizontal geodesic in $\mathcal{C} \cap X_\beta$ which connects the two v-vertical trees through x and y. Assume that the endpoints of h are exponentially separated after t_0 in the direction of $[\beta, \alpha]$. Then:

$$\lambda^n |I_D|_{hor} \leq |h|_{hor} \leq \lambda^{-n}(|I_D|_{hor} + |I_C|_{hor}) \tag{1}$$

so that

$$|I_C|_{hor} \geq \frac{\lambda^n - \lambda^{-n}}{\lambda^{-n}} |I_D|_{hor}$$

and consequently, since $d_{hor}(x,y) = |I_D|_{hor} + |I_C|_{hor}$,

$$|I_C|_{hor} \geq \frac{X(n)}{1 + X(n)} d_{hor}(x,y)$$

with $X(n) = \frac{\lambda^n - \lambda^{-n}}{\lambda^{-n}}$. Since $\lim\limits_{n \to +\infty} \dfrac{X(n)}{1 + X(n)} = 1$, there is $n_\star \geq 0$ such that for any $n \geq n_\star$,

$$|I_C|_{hor} \geq \frac{1}{2} d_{hor}(x,y).$$

But, by definition, the horizontal length of each subgeodesic in I_C is smaller than Cte. Thus the number of elements in I_C is at least the integer part of $\frac{1}{2Cte} d_{hor}(x,y) + 1$. Furthermore, since g is a v-telescopic chain, the telescopic length of any subset of g containing j horizontal geodesics is at least $(j-1)$. We so obtain:

$$|g|^v_{tel} \geq \frac{1}{2Cte} d_{hor}(x,y).$$

On the other hand:

$$d^v_{tel}(x,y) \leq \lambda^{-n} d_{hor}(x,y) + 2nt_0.$$

since there is a v-telescopic chain between x and y the telescopic length of which is given by the right-hand side of the above inequality. Since g is a (a, b)-quasi geodesic, the last two inequalities give $n_{\star\star} \geq 0$ such that for $n \geq n_{\star\star}$:

$$d_{hor}(x, y) \leq \frac{2ant_0 + b}{\frac{1}{2Cte} - a\lambda^{-n}}.$$

Taking the maximum of $n_\star, n_{\star\star}$ and the above upper-bound for $d_{hor}(x, y)$, we get the announced constant in the case where the endpoints of the horizontal geodesic h above are exponentially separated in the direction of $[\beta, \alpha]$. If not, there are in all the other directions so that we easily get a constant $N \geq 0$ such that $d_{hor}(\omega x, \omega y) \geq \lambda d_{hor}(x, y)$ for any $\mathcal{T}$-geodesic ω with $|\omega|_{\mathcal{T}} = Nt_0$ and $[\pi(x), \pi(h)] \subset \omega$. Lemma 8.2 is then easily deduced. $\square$

As a consequence we have:

Corollary 8.3. *For any $a \geq 1, b \geq 0$ and $d \geq M$, there exists $C \geq d$ such that if $\mathcal{C}$ is a generalized w-corridor with exponentially separated v-vertical segments, if g is any v-telescopic chain which is a (a, b)-quasi geodesic of $(\mathcal{C}, d^v_{tel})$, if x, y are the endpoints of two v-vertical segments s, s' over a same edge-path in $\mathcal{T}$, with $\pi(s) \cap \pi(g) = \{\alpha\}$ and such that $d_{hor}(s, s') \leq d$, then $d_{hor}(x, y) \leq C$.*

Remark 8.4. At this point, we would like to notice that Lemma 8.2 is similar to Lemma 6.7 of [10]. However in addition of some misprints, a slight mistake took place there in the proof of the Lemma. Indeed the inequality (1) in the proof of Lemma 8.2 is true here, in the generalized corridor, but there the constant λ should have been modified to take into account the so-called “cancellations”.

Lemma 8.5. *For any $r \geq 0$, there exists $C \geq 0$ such that if $\mathcal{C}$ is a generalized w-corridor with exponentially separated v-vertical segments, if x and y are the endpoints of a r-vertical segment s in $\mathcal{C}$, if the intersection-point z of some v-vertical tree through y in $\mathcal{C}$ with the stratum $X_{\pi(x)}$ satisfies $d_{hor}(x, z) \geq C$, then for any $\mathcal{T}$-geodesic ω with $|\omega|_{\mathcal{T}} = nt_0$, $n \geq 1$, and $\omega \cap \pi(s) = \{\pi(x)\}$, $d_{hor}(\omega x, \omega z) \geq \lambda^n d_{hor}(x, z)$.*

Proof. If $|s|_{vert} \leq t_0$, the existence of the constants of quasi isometry, item (a) of Lemma 3.3, and the definition of a r-vertical segment give an upper-bound for $d_{hor}(x, z)$. Let us thus assume $|s|_{vert} > t_0$. Choose d such that $\lambda d - r' \geq 2r'$, where r' is the above upper-bound when $|s|_{vert} = t_0$. Then set $C = \max(d, M)$. Assume that $d_{hor}(x, z) \geq C$ and that x and z are exponentially separated in the direction given by s. If $[\pi(x), \pi(y)] = \omega_0\omega'$ with $|\omega_0|_{\mathcal{T}} = t_0$, then $d_{hor}(\omega_0 x, \omega_0 z) \geq \lambda d_{hor}(x, z)$. Thanks to the inequality used to define d, one easily concludes that the horizontal distance between s and the vertical tree through y increases along s when going from x to y which of course cannot happen. The conclusion follows from the exponential-separation property. $\square$

Proof of Proposition 4.7. We are given a w-corridor $\mathcal{C}$, L the horizontal distance between two points x and y in $\mathcal{C}$, and g a (a, b)-quasi geodesic in $(\mathcal{C}, d^v_{tel})$ from a v-vertical tree through x to a v-vertical tree through y with $v \geq C_{4.4}(w)$ ($C_{5.3}(v)$ in the case of a generalized corridor). We assume that the v-vertical segments in $\mathcal{C}$ are exponentially separated. We consider the region R with vertical width $C_{8.3}$ centered at the stratum X_α with $\alpha = \pi(x)$. We decompose g in three subsets: the first one, denoted g_0, from the initial point of g until the first point z in $g \cap R$, the second one, denoted g_1, from z to the last point t in $g \cap R$, the third one, denoted g_2, from t to the terminal point of g. Obviously g_1 can be approximated by the concatenation of two vertical segments with a horizontal geodesic in X_α (the approximation constant only depend on L, a and b). We denote by g_1' the resulting set.

We now consider a maximal chain in g_0 which satisfies the following properties:

- its endpoints lie in a same stratum X_β,
- its image under π does not intersect $[\alpha, \beta)$.

From Corollary 8.3, the endpoints of such a subchain are at horizontal distance smaller than $C_{8.3}$ one to each other. Thus, by substituting each such subchain by a horizontal geodesic connecting its endpoints, we construct a $C_{8.3}$-vertical segment g_0'. We do the same thing for g_2, so obtaining a $C_{8.3}$-vertical segment g_2'. From Lemma 8.5, $g' = g_0' \cup g_1' \cup g_2'$ lies in a bounded neighborhood of the v-vertical segments connecting its endpoints to x_1 and x_2. From the construction, $d_{tel}^H(g, g') \leq aC_{8.3} + b + 1$. The proposition follows. $\square$

9. Quasiconvexity of corridors

In this section we prove Proposition 4.8, its adaptation to generalized corridors and Proposition 7.4.

9.1. **Two basic lemmas.** We need first a very general lemma about Gromov hyperbolic spaces.

Lemma 9.1. *Let (X, d) be a Gromov hyperbolic space. There exists $C \geq 0$ such that for any $r \geq C$ there is $D \geq 0$, increasing and affine in r, such that if $[x, y]$ is a diameter of a ball $B_{x_0}(r)$, if ω is any chain in X with $\omega \cap B_{x_0}(r) = \{x, y\}$, then $|\omega|_d \geq e^D$.*

This lemma is a rewriting of Lemma 1.6 of [7]. $\square$

Lemma 9.2. *Let $\widetilde{X}$ be a tree of δ-hyperbolic spaces which satisfies the exponential-separation property. For any $v \geq C_{3.6}$, there exists $C \geq 0$ such that if x, y, z, t are the vertices of a geodesic quadrilateral in some stratum X_α, with $d_{hor}(x, z) \leq 2\delta$, $d_{hor}(y, t) \leq 2\delta$, and $d_{hor}(x, y) \geq C$, $d_{hor}(z, t) \geq C$, then for any $\mathcal{T}$-geodesic ω with $|\omega|_{\mathcal{T}} \geq C_{3.7} + nt_0$ and starting at $\pi(x)$, when considering the v-vertical segments over ω we have:*

$$d_{hor}(\omega x, \omega y) \geq \lambda^n d_{hor}(x, y) \Leftrightarrow d_{hor}(\omega z, \omega t) \geq \lambda^n d_{hor}(z, t)$$

Proof. If A, B are two subsets of a metric space (X, d), we set $d^s(A, B) = \sup_{x \in A, y \in B} d(x, y)$. Let us consider any $\mathcal{T}$-geodesic ω with $|\omega|_{\mathcal{T}} = t_0$ starting at α. From Lemma 3.3,

$$d^s_{hor}(\omega x, \omega z) \leq \lambda_+^{t_0}(2\delta + \mu)$$

and

$$d^s_{hor}(\omega y, \omega t) \leq \lambda_+^{t_0}(2\delta + \mu).$$

Assume $d_{hor}(\omega x, \omega y) \geq \lambda d_{hor}(x, y)$ but $d_{hor}(\omega z, \omega t) < \lambda d_{hor}(z, t)$.

We take $d_{hor}(x, y) \geq M$ and $d_{hor}(z, t) \geq M$. Assume $d^s_{hor}(\omega z, \omega t) \leq \frac{1}{\lambda} d_{hor}(z, t)$. But $d_{hor}(z, t) \leq 4\delta + d_{hor}(x, y)$. Putting together these inequalities we get

$$\lambda d_{hor}(x, y) \leq 2\lambda_+^{t_0}(2\delta + \mu) + \frac{1}{\lambda}(4\delta + d_{hor}(x, y)).$$

Whence an upper bound for $d_{hor}(x, y)$ and thus for $d_{hor}(z, t)$. If $d^s_{hor}(\omega z, \omega t) > \frac{1}{\lambda} d_{hor}(z, t)$ then the lemma follows from the definition of the constant $C_{3.7}$, see the corresponding lemma. $\square$

The above two lemmas are not needed if one only considers trees of 0-hyperbolic spaces, the proof in this last case being much simpler.

9.2. **Approximation of quasi geodesics with bounded vertical deviation.**

Lemma 9.3 below states that in a tree of hyperbolic spaces $(\widetilde{X}, \mathcal{T})$ a quasi geodesic with bounded image in $\mathcal{T}$ lies close to a corridor between its endpoints. This is intuitively obvious and nothing is new neither surprising in the arguments of the proof: they heavily rely upon the δ-hyperbolicity of the strata and the fact that strata are quasi isometrically embedded into each other. For the sake of brevity, we do not develop them here.

Lemma 9.3. *Let $(\widetilde{X}, \mathcal{T}, \pi)$ be a tree of hyperbolic spaces. For any $\kappa, b \geq 0, a \geq 1$ and $v \geq C_{3.6}$ there exists $C \geq 0$ such that if g is any (a,b)-quasi geodesic of $\widetilde{X}$ with* $\mathrm{diam}_{\mathcal{T}}(\pi(g)) \leq \kappa$*, if $\mathcal{C}$ is a generalized v-corridor whose vertical boundaries pass through the endpoints of g then $g \subset \mathcal{N}^C_{\widetilde{X}}(\mathcal{C})$.*

9.3. **Stairs. Notations:** The sign $\simeq_1$ stands for an equality up to ± 1, $(\widetilde{X}, \mathcal{T}, \pi)$ a tree of hyperbolic spaces which satisfies the exponential-separation property, $v \geq C_{3.6}$ a constant.

Definition 9.4. Let $r \geq M$. A *r-stair relative to a generalized v-corridor $\mathcal{C}$* is a v-telescopic chain $\mathcal{S}$ the vertical segments of which have vertical length greater than $C_{3.7}$ and such that, for any horizontal geodesic $[a_i, b_i]$ in $\mathcal{S}$:

(a) $d_{hor}(a_i, b_i) \geq r$ and $d_{hor}([a_i, b_i], \mathcal{C}) \simeq_1 d_{hor}(a_i, P^{hor}_{\mathcal{C}}(a_i))$,
(b) any two points $a, b \in [a_i, b_i]$ with $d_{hor}(a, b) \geq r$ are exponentially separated in the direction of the $\mathcal{T}$-geodesic $[\pi(a_i), \pi(a_{i+1})]$.

Lemma 9.5. *With the notations of Definition 9.4: there exist $C \geq C_{9.2}$ such that for any $r \geq C$, if $\mathcal{C}$ is a generalized v-corridor, if $\mathcal{S}$ is a r-stair relative to $\mathcal{C}$, if $\mathcal{U}$ is a generalized v-corridor between a vertical tree through the terminal point of $\mathcal{S}$ and a vertical boundary of $\mathcal{C}$, then*

$$\mathcal{S} \subset \mathcal{N}^{r+2\delta}_{hor}(\mathcal{U}).$$

Proof. Let $a_i, b_i \in \mathcal{S}$ as given in Definition 9.4 and let z be a point at the intersection of the stratum $X_{\pi(a_i)}$ with a vertical tree through some point farther in the stair. Then:
Claim 1: There exists $K > 0$ not depending on a_i nor z such that, if r is sufficiently large then $d_{hor}([a_i, z], \mathcal{C}) \geq d_{hor}(a_i, P^{hor}_{\mathcal{C}}(a_i)) - K$.
Proof of Claim 1: Choose K such that $e^{D_{9.1}(K)} > 4\delta + 1$ and assume $d_{hor}([a_i, z], \mathcal{C}) < d_{hor}(a_i, P^{hor}_{\mathcal{C}}(a_i)) - K$. Then Lemma 9.1 implies that $[b_i, z]$ descends at least until a 2δ-neighborhood of a_i. Assume $r \geq C_{9.2} + 2\delta$. Then Lemma 9.2 gives an initial segment of $[b_i, z]$ of horizontal length greater than $r - 2\delta$ which is dilated in the direction of $[\pi(a_i), \pi(a_{i+1})]$. If r is chosen sufficiently large with respect to the constants of hyperbolicity for a corridor, we get z' at the intersection of the considered vertical tree through z with the stratum $X_{\pi(a_{i+1})}$ such that $d_{hor}([a_{i+1}, z'], \mathcal{C}) < d_{hor}(a_{i+1}, P^{hor}_{\mathcal{C}}(a_{i+1})) - K$. The repetition of these arguments show that the horizontal distance between $\mathcal{S}$ and the vertical tree through z does not decrease along $\mathcal{S}$. This is an absurdity since z was chosen in a vertical tree through a point farther in $\mathcal{S}$. The proof of Claim 1 is complete.
Claim 2: There exists $K(r)$ not depending on b_i nor z such that, if r is sufficiently large then $d_{hor}([b_i, z], \mathcal{C}) \geq d_{hor}(b_i, P^{hor}_{\mathcal{C}}(b_i)) - K(r)$.
Proof of Claim 2: Let $z_\star \in [b_i, z]$ with $d_{hor}(z_\star, P^{hor}_{\mathcal{C}}(z_\star)) \simeq_1 \max(d_{hor}([b_i, z], \mathcal{C}), d_{hor}(a_i, P^{hor}_{\mathcal{C}}(a_i)))$. From the δ-hyperbolicity of the strata, $[b_i, z_\star]$ lies in the horizontal 2δ-neighborhood of $[a_i, b_i]$. Assume $d_{hor}(b_i, z_\star) \geq r$ and is sufficiently large to apply Lemma 9.2. Then there is $K(r)$ such that, if $z_\star$ satisfies $d_{hor}(z_\star, P^{hor}_{\mathcal{C}}(z_\star)) < d_{hor}(b_i, P^{hor}_{\mathcal{C}}(b_i)) - K(r)$, the points b_i and $z_\star$ are exponentially separated in the direction of $[\pi(a_i), \pi(a_{i+1})]$. We thus obtain at a_{i+1} a situation similar to that of Claim 1. The proof of Claim 2 follows.

Lemma 9.5 is easily deduced from the above two claims, we leave the reader work out the easy details. □

Lemma 9.6. *For any $r \geq C_{9.5}$ there exists $C > 0$ such that, if $\mathcal{C}$ is a generalized v-corridor, if $\mathcal{S}$ is a r-stair relative to $\mathcal{C}$ which is not contained in the vertical C-neighborhood of the stratum containing its initial point, then the terminal point of $\mathcal{S}$ does not belong to the r-neighborhood of $\mathcal{C}$ in $\widetilde{X}$.*

Proof. Decompose $\mathcal{S}$ in maximal substairs $\mathcal{S}_0 \cdots \mathcal{S}_k$ such that $\pi(\mathcal{S}_j)$ is a geodesic of $\mathcal{T}$. Let $[a_i, b_i]$ be the first horizontal geodesic in $\mathcal{S}_j$, let x be the initial point of $\mathcal{S}_j$ and let z be any point in $\mathcal{S}_j$ with $nt_0 \leq d_{\mathcal{T}}(\pi(z), \pi(x)) \leq (n+1)t_0$.

The inequality

$$d_{hor}(z, P^{hor}_{\mathcal{C}}(z)) \geq Cte\lambda^n d_{hor}(a_i, b_i) \tag{2}$$

is an easy consequence of the definition of a stair and of Lemma 9.2 as soon as $r \geq C_{9.2}$. Indeed, the initial segment of horizontal length r in $[b_i, P^{hor}_{\mathcal{C}}(b_i)]$ lies in the horizontal 2δ-neighborhood of $[b_i, a_i]$. The assertion then follows from item (b) of Definition 9.4 and Lemma 9.2.

The inequality (2) readily gives the announced result. □

9.4. **Approximation of a quasi geodesic by a stair.**
Notations: $(\widetilde{X}, \mathcal{T})$ a tree of δ-hyperbolic spaces which satisfies the exponential-separation property, $v \geq C_{3.6}$.

Lemma 9.7. *For any $a \geq 1, b \geq 0$ there exists $D \geq 0$ such that for any $r \geq D$ there are $C, E \geq 0$, where E is affine in r, such that if $\mathcal{C}$ is a generalized v-corridor, if the endpoints of a v-telescopic (a,b)-quasi geodesic g are in a horizontal r-neighborhood of $\mathcal{C}$, if g lies in the closed complement of this horizontal neighborhood and if the vertical segments in g have vertical length greater than $3(C_{3.7} + D_{8.1})$ then either g lies in the C-neighborhood of a E-stair relative to $\mathcal{C}$ or g is contained in the C-neighborhood of $\mathcal{C}$.*

Proof. We decompose the proof in two steps. The first one is only a warm-up, to present the ideas in a particular, but important, case. The general case, detailed in the second step, is technically more involved but no new phenomenon appears.

Step 1: Proof of Lemma 9.7 when the horizontal length of any horizontal path in g is greater than some constant (depending on a et b). The endpoints of any horizontal path h in g with horizontal length greater than $C_{8.1}$ are exponentially separated under every geodesic ω of $\mathcal{T}$ with length $D_{8.1}$. If $|h|_{hor} \geq C_{9.2}$, this is also true for any horizontal geodesic h' in the 2δ-neighborhood of h. Finally, if $|h|_{hor}$ is sufficiently large, by the exponential-separation property, the endpoints of h are also exponentially separated in any v-corridor containing h. If $e(a,b)$ (we do not indicate the dependance on v) is the maximum of the above constants, we now assume $|h|_{hor} \geq 3e(a,b)$.

Let us consider two consecutive horizontal geodesics h_1, h_2 in g, separated by a vertical segment s. Let $\mathcal{D}$ be a corridor containing h_1 and s. Then:

$$|h_2 \cap \mathcal{N}^{2\delta}_{hor}(\mathcal{D})|_{hor} \leq e(a,b). \tag{3}$$

Otherwise we have a contradiction with the fact that the endpoints of any subgeodesic of h_2 whose length is greater than $C_{8.1}$ are exponentially separated in the direction of h_1.

From the inequality (3), the concatenation of h_1, s and h_2 is $e(a,b)$-close, with respect to the horizontal distance, of a $2e(a,b)$-stair relative to $\mathcal{C}$ if $d_{hor}(h_1, \mathcal{C}) \simeq_1 d_{hor}(a_1, P^{hor}_{\mathcal{C}}(a_1))$ where a_1 is the initial point of h_1.

Let us now set $r \geq 3e(a,b)$ and assume that the horizontal geodesics in g have horizontal length greater than r. Let x be the initial point of g (in particular $d_{hor}(x, P^{hor}_{\mathcal{C}}(x)) \simeq_1 r$). Let s be the vertical segment starting at x and ending at y in g. Let h be the horizontal geodesic following s along g. Let $n \geq 1$ be the greatest integer with $n(C_{3.7}+D_{8.1}) \leq |s|_{vert}$.

By assumption x and $P^{hor}_{\mathcal{C}}(x)$ are exponentially separated in the direction of s. Since the strata are quasi isometrically embedded one into each other, this gives $\kappa > 1$ such that, any two points $p, q \in [x, P^{hor}_{\mathcal{C}}(x)]$ with $d_{hor}(p,q) \geq \max(\frac{1}{\kappa}r, M)$ satisfy $d_{hor}(\pi(s)p, \pi(s)q) \geq \lambda^n d_{hor}(p,q)$. Thus the same arguments as those exposed above when working with h_1, h_2 show that $|h \cap \mathcal{N}^{2\delta}_{hor}([y, P^{hor}_{\mathcal{C}}(y)])|_{hor} \leq \max(e(a,b), \frac{1}{\lambda^n \kappa}r, M)$. If n is greater than some critical constant n_*, this last maximum is equal to $e(a,b)$. Thus, in this case we take $h_1 = [x, P^{hor}_{\mathcal{C}}(x)]$ and $h_2 = h$: the above arguments prove that the concatenation of h_1, s and h_2 is $e(v,a,b)$-close to a $e(a,b)$-stair. If n is smaller than n_*, then we substitute r by $\lambda_+^{n_*(C_{3.7}+D_{8.1})} r$, modify g by taking the starting point at the endpoint y of s and take h_1 as the first horizontal geodesic.

In both cases, by repeating the arguments above at any two consecutive horizontal geodesic following the first two ones along g, we show that g is $e(a,b)$-close, with respect to the horizontal distance, of a $e(a,b)$-stair relative to $\mathcal{C}$. □

Step 2: Adaptation of the argument to the general case: The boundary trees of $\mathcal{C}$ are denoted by L_1 and L_2, and g goes from L_1 to L_2. We choose a positive constant r, which when necessary will be set sufficiently large with respect to the constants $C_{9.5}$, M, δ and $C_{9.2}$. Let x_0 be the initial point of g. It lies in the boundary of the horizontal r-neighborhood of $\mathcal{C}$. We denote by $\mathcal{C}_i$ and x_i, $i = 1, \cdots$, a sequence of corridors and points of g defined inductively as follows:

(a) $\mathcal{C}_i$ is a corridor with boundary trees a v-vertical tree through x_{i-1} and the v-vertical boundary L_2 of $\mathcal{C}$,

(b) x_i is the first point following x_{i-1} along g such that $d_{hor}(x_i, P^{hor}_{\mathcal{C}_i}(x_i)) \geq r$.

The chain in g between x_{i-1} and x_i is denoted by $g_{i-1,i}$. Obviously $g_{i-1,i}$ is contained in the horizontal r-neighborhood of $\mathcal{C}_i$. We project it to $\mathcal{C}_i$. From Lemma 4.10, we get a $D_{4.10}$-telescopic $(C_{4.10}, C_{4.10})$-quasi geodesic of $(\mathcal{C}_i, d^{D_{4.10}}_{tel})$. We set $X(a,b,r) = C_{4.7}(r, C_{4.10}, C_{4.10})$. From Proposition 4.7, $P^{hor}_{\mathcal{C}_i}(g_{i-1,i})$ is contained in the $X(a,b,r)$-neighborhood of the concatenation of a subpath of $[x_{i-1}, P^{hor}_{\mathcal{C}_{i-1}}(x_{i-1})]$ with a vertical segment in $\mathcal{C}_i$ (and is followed by $[P^{hor}_{\mathcal{C}_i}(x_i), x_i]$). Consider in this approximation of (a subchain of) g a maximal collection of points y_i which defines a r-stair relative to $\mathcal{C}$. The points y_i do not necessarily agree with the x_i's, because it might happen that, after x_{i-1} for instance, the approximation constructed above reenters in the r-neighborhood of $\mathcal{C}_{i-1}$ before leaving the r-neighborhood of $\mathcal{C}_i$. We proceed as in Step 1 and choose the y_i's so that:

(a) either y_i is contained in a horizontal geodesic of the chain, and from the observations in Step 1, this horizontal geodesic may be included in a stair,

(b) or the vertical distance from y_i to the next horizontal geodesic is at least $C_{3.7}+D_{8.1}$.

Either we obtain a non-trivial r-stair relative to $\mathcal{C}$ which approximates a subchain g'_0 of g or the approximation we constructed above exhausts g and is contained in some telescopic neighborhood of $\mathcal{C}$ the size of which is obtained from the previously exhibited constants. In this last case, the same assertion holds for the whole g. This is one of the announced alternatives.

We can thus assume that we got $y_0, \cdots, y_k$ forming a r-stair relative to $\mathcal{C}$. It is denoted by S. Since the strata are quasi isometrically embedded one into each other, there is

$\kappa > 1$, only depending on the constants of quasi isometry, such that S is in fact a $\max(\frac{1}{\kappa}r, M, e(a,b))$-stair relative to $\mathcal{C}$. As soon as $r > \kappa(M + e(a,b))$, which we suppose from now, this maximum is just $\frac{1}{\kappa}r$. Thus S is a $\frac{r}{\kappa}$-stair whose horizontal geodesics have horizontal length at least r.

By construction S approximates $g'_0 \subset g$. We now consider the maximal subchain g'_1 of g starting at (or near - recall that we constructed an approximation of a subchain of g) y_k which lies in the r-neighborhood of $\mathcal{C}_k$. This last corridor plays the rôle of the corridor $\mathcal{U}$ of Lemma 9.5. We project the subchain g'_1 to $\mathcal{C}_k$, so getting a $(C_{4.10}, C_{4.10})$-quasi geodesic of this corridor. From Lemma 9.5, and because of the hyperbolicity of the strata, each horizontal geodesic of the $\frac{r}{\kappa}$-stair S admits a subgeodesic with horizontal length greater than $\frac{\kappa-1}{\kappa}r$ in the horizontal 2δ-neighborhood of $\mathcal{C}_k$. If r is chosen sufficiently large, Lemma 9.2 gives horizontal geodesics in $\mathcal{C}_k$ with horizontal length greater than M which are dilated in the same directions than the horizontal geodesics of S. Now Proposition 4.7 applies and allows us to approximate the projection of g'_1 on $\mathcal{C}_k$ by a sequence of these horizontal geodesics. But each one of these horizontal geodesics is close to a point in $g'_0 \subset g$. Thus, since g is a (a,b)-quasi geodesic, the vertical length of g'_1, and so its telescopic length, is bounded from above by a constant depending on a and b. So we can forget g'_1 and continue the construction of our $\frac{r}{\kappa}$-stair relative to $\mathcal{C}$ at the point where the approximation of g'_1 leaves the r-neighborhood of $\mathcal{C}_k$. We eventually exhaust g and obtain a $\frac{r}{\kappa}$-stair relative to $\mathcal{C}$. □

9.5. **Proof of Proposition 4.8.** Let g and $\mathcal{C}$ be as given by this proposition. Assume that some subchain g' of g leaves and then reenters the horizontal $D_{9.7}$-neighborhood of $\mathcal{C}$. Assume that g' is not contained in the telescopic $C_{9.7}(D_{9.7}, a, b)$-neighborhood of $\mathcal{C}$. We set $C_{9.7} \equiv C_{9.7}(D_{9.7}, a, b)$ and $E_{9.7} \equiv E_{9.7}(D_{9.7}, a, b)$.

Suppose for the moment that the vertical segments in g' have vertical length greater than $3(C_{3.7} + D_{8.1})$. Then Lemma 9.7 gives G, a $E_{9.7}$-stair relative to $\mathcal{C}$ with $d^H_{tel}(g', G) \leq C_{9.7}$. From Lemma 9.6, G does not leave the vertical $C_{9.6}(E_{9.7})$-neighborhood of the stratum containing the initial point of G. Therefore, by setting $V(a,b) = C_{9.6}(E_{9.7}) + C_{9.7}$, g' does not leave the vertical $V(a,b)$-neighborhood of this stratum. From Lemma 9.3, g' lies in the telescopic $C_{9.3}(V(a,b), a, b)$-neighborhood of $\mathcal{C}$.

It remains to consider the case where the vertical segments in g' are not sufficiently large. Let s be a vertical segment in g with $|s|_{vert} < X \equiv 3(C_{3.7} + D_{8.1})$.

(†) Thanks to the assumption that all the attaching-maps of the tree of hyperbolic spaces are quasi isometries, s is contained in a vertical segment s' of vertical length greater than X. We modify g' by sliding, along s', a horizontal geodesic in g' incident to s until getting a vertical segment with vertical length X. This yields a new telescopic (a', b')-quasi geodesic in a bounded neighborhood of g, where the constants a', b' only depend on a, b and on the constants of quasi isometry. After finitely many such moves, we obtain a quasi geodesic as desired, and we are done. Since the vertical distance between two strata is uniformly bounded away from zero, after finitely many such substitutions, we eventually get a quasi geodesic, in a bounded neighborhood of g, which satisfies the assumptions required by Lemma 9.7. This completes the proof of Proposition 4.8. □

9.6. **Adaptation to generalized corridors.** The only problem is to get a telescopic chain with vertical segments sufficiently large. We start from the sentence marked by a (†) in the preceding subsection. If s is not contained in a vertical segment s' of vertical length greater than X, we obtain a vertical segment $\mathbf{s}$ from b_i to a_{i+1} satisfying the following properties (we still denote by g' the (a', b')-quasi geodesic eventually obtained, we denote by $\mathbf{s}_0$ the vertical segment of g' ending at a_i and by $\mathbf{s}_1$ the one starting at b_{i+1}):

(a) there is no vertical segment starting at a_i (resp. at a_{i+1}) over the edge $\pi(\mathbf{s})$ (resp. over $\pi(\mathbf{s}_1)$);

(b) there is no vertical segment ending at b_i over $\pi(\mathbf{s}_0)$.

Consider horizontal geodesics $\alpha_i = [a_i, P^{hor}_{\mathcal{C}}(a_i)]$, $\beta_i = [b_i, P^{hor}_{\mathcal{C}}(b_i)]$, $\alpha_{i+1} = [a_{i+1}, P^{hor}_{\mathcal{C}}(a_{i+1})]$ and $\beta_{i+1} = [b_{i+1}, P^{hor}_{\mathcal{C}}(b_{i+1})]$. By the δ-hyperbolicity of the strata, there is $a'_i \in [a_i, b_i] \cap \mathcal{N}^{2\delta}_{hor}(\alpha_i \cup \beta_i)$ and $b'_i \in [a_{i+1}, b_{i+1}] \cap \mathcal{N}^{2\delta}_{hor}(\alpha_{i+1} \cup \beta_{i+1})$. Because the strata are quasi isometrically embedded one into each other, we get two points a''_i, b''_i which satisfy:

(A) they are Y-close (with respect to the horizontal distance) respectively to a'_i and b'_i, where the constant Y only depends on δ and on the constants of quasi isometry;

(B) there is a v-vertical segment from a''_i to b''_i which is contained in a larger v-vertical segment going over $\pi(\mathbf{s}_0)$ and $\pi(\mathbf{s}_1)$.

We modify g' by going from a_i to a''_i then to b''_i and eventually end at b_{i+1}. The resulting chain is a (a'', b'')-quasi geodesic, where the constants a'', b'' only depends on δ and on the constants of quasi isometry. Moreover this new chain is in a bounded neighborhood of g'. Thanks to item (B), we can modify it by enlarging the vertical segment from a''_i to b''_i. The conclusion in then the same as in the preceding subsection. □

9.7. **Proof of Proposition 7.4.** The arguments are similar to those exposed for proving the quasi convexity of the corridors. We give here only a sketch of the proof. Because a tree of cone-vertices is a vertical tree, the horizontal deviation of a tree of cone-vertices with respect to $\mathcal{C}$ depends linearly on the vertical variation of the orbit. Thus, if a sufficiently large segment of the orbit remains outside a sufficiently large horizontal neighborhood of $\mathcal{C}$, the exponential separation implies that the horizontal distance between the orbit and $\mathcal{C}$ exponentially increases with the vertical length of the orbit. Assume now that the exceptional orbit considered is followed by another one. The *strong* exponential-separation property gives the same consequence: this second exceptional orbit does not go back to $\mathcal{C}$ and the horizontal distance with respect to $\mathcal{C}$ exponentially increases with its vertical length, as soon as this length is sufficiently large. Here the arguments are similar to those used for proving Lemmas 9.5 and 9.6. Finally, if the exceptional orbit is followed by a quasi geodesic in $\widehat{X}$, then the approximation by a stair as was done before, yields the same conclusion. □

References

[1] E. ALIBEGOVIC 'A combination theorem for relatively hyperbolic groups', *Bulletin of the London Mathematical Society* (3) 37 (2005) 459–466.

[2] M. BESTVINA and M. FEIGHN 'A combination theorem for negatively curved group' *Journal of Differential Geometry* (1) 35 (1992) 85–101. With an addendum and correction *Journal of Differential Geometry* (4) 43 (1996) 783–788.

[3] B.H. BOWDITCH 'Relatively hyperbolic groups', preprint, University of Southampton 1999 (http://www.warwick.ac.uk/∼masgak/preprints.html).

[4] B.H. BOWDITCH 'Stacks of hyperbolic spaces and ends of 3-manifolds', preprint, University of Southampton 2002 (http://www.warwick.ac.uk/∼masgak/preprints.html).

[5] M. BRIDSON and A. HAEFLIGER *Metric spaces of non-positive curvature* Grundlehren der Mathematischen Wissenschaften 319, Springer-Verlag (1999).

[6] I. BUMAGIN 'On definitions of relatively hyperbolic groups', *Geometric methods in group theory* Contemporary Mathematics series 372 (2005).

[7] M. COORNAERT, T. DELZANT and A. PAPADOPOULOS *Géométrie et théorie des groupes* Lecture Notes in Math. 1441, Springer Verlag (1990).

[8] F. DAHMANI 'Combination of convergence groups' *Geometry and Topology* 7 (2003) 933–963.

[9] B. FARB 'Relatively hyperbolic groups' *Geom. Funct. Anal. (GAFA)* 8 (1998) 1–31.

[10] F. GAUTERO 'Hyperbolicity of mapping-torus groups and spaces' *L'Enseignement mathématique*, 49 (2003) 263–305.

[11] F. GAUTERO *Quatre problèmes dynamiques, géométriques ou algébriques autour de la suspension.* Habilitation thesis, Université Blaise Pascal, Clermont-Ferrand (2006).

[12] F. GAUTERO and M. LUSTIG 'Relative hyperbolization of (one-ended hyperbolic)-by-cyclic groups' *Math. Proc. Camb. Phil. Soc.* 137 (2004) 595–611.

[13] R. GITIK 'On the combination theorem for negatively curved groups' *International Journal of Algebra Comput.* (6) 6 (1996) 751–760.

[14] M. GROMOV 'Hyperbolic groups', *Essays in Group Theory* Math. Sci. Res. Inst. Publ. 8, Springer 1987, 75–263.

[15] I. KAPOVICH 'A non-quasiconvexity embedding theorem for hyperbolic groups' *Mathematical Proceedings of the Philosophical Cambridge Society* 127 (1999) 461–486.

[16] I. KAPOVICH 'Mapping tori of endomorphisms of free groups' *Communications in Algebra* (6) 28 (2000) 2895–2917.

[17] O. KHARLAMPOVICH and A. MYASNIKOV 'Hyperbolic groups and free constructions' *Transactions of the American Mathematical Society* (2) 350 (1998) 571–613.

[18] M. MJ and L. REEVES 'A combination theorem for strong relative hyperbolicity', *Geometry and Topology* 12 (3) (2008) 1777-1798.

[19] D. OSIN 'Weak hyperbolicity and free constructions' *Contemporary Mathematics* 360 (2004) 103–111.

[20] D. OSIN 'Relatively hyperbolic groups: Intrinsic geometry, algebraic properties and algorithmic problems' *Memoirs of the American Mathematical Society* (843) 179 (2006).

[21] D. OSIN 'Relative Dehn functions of amalgated products and HNN extensions' *Contemporary Mathematics* 394 (2006) 209–220.

[22] A. SZCZEPAŃSKI 'Relatively hyperbolic groups' *Michigan Math. Journal* 45 (1998) 611–618.

Université de Nice Sophia Antipolis, Parc Valrose, Laboratoire de Mathématiques J.A. Dieudonné, UMR CNRS 7351, 06108 Nice Cedex 02, France

E-mail address: `Francois.Gautero@unice.fr`